On Approximate Cloaking by Nonsingular Transformation Media

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Geometric Analysis on Euclidean and Homogeneous Spaces

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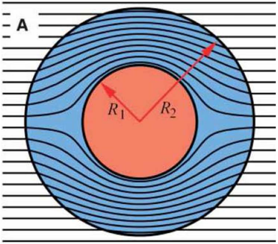
Invisibility and Cloaking



From Pendry et al's paper

- J. B. Pendry, D. Schurig and D. R. Smith (2006)
- U. Leonhard (2006)
- Transformation Optics and *Metamaterials*
- A. Greenleaf, M. Lassas and G. Uhlmann (2003)

Outline

1. Cloaking for Electrostatics

2. On Electromagnetic Cloaking (joint work with Hongyu Liu)
   - Ideal cloaking for Maxwell's equations
   - Nonsingular approximate cloaking
     - Regularization scheme
     - Cloaking a passive medium
     - Cloaking an active medium
     - Cloak-busting inclusion and lossy layer

3. On 2D Acoustic Cloaking (joint work with Matti Lassas)
   - Ideal cloaking for the Helmholtz equation
   - Nonsingular regularization and non-local boundary conditions

Outline

Non-uniqueness for Calderón's problem

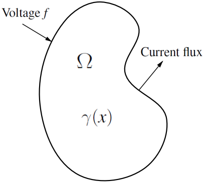
Conductivity equation:

$$\nabla \cdot \gamma \nabla u = 0 \quad \text{in } \Omega.$$

Dirichlet to Neumann map:

$$\Lambda_\gamma : f \mapsto \nu \cdot \gamma \nabla u|_{\partial\Omega}.$$

Anisotropic conductivity:
$\gamma = (\gamma^{ij})$ pos. def. sym. matrix



Calderón's problem: $\Lambda_{\gamma_1} = \Lambda_{\gamma_2} \Rightarrow \gamma_1 = \gamma_2$?

Non-uniqueness

If $\psi : \Omega \to \Omega$ is a diffeomorphism, $\psi|_{\partial\Omega} =$ Identity, then

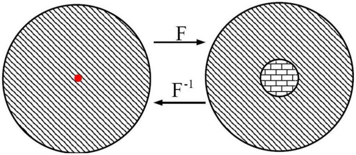
$$\Lambda_{\psi_*\gamma} = \Lambda_\gamma \text{ where } \psi_*\gamma = \left(\frac{(D\psi)^T \gamma \, (D\psi)}{|\det\,(D\psi)|} \right) \circ \psi^{-1}.$$

Cloaking for EIT

$$F : B_2 \backslash \{0\} \to B_2 \backslash \overline{B_1}$$

$$F(y) = \left(1 + \frac{|y|}{2}\right) \frac{y}{|y|}.$$

$$F|_{\partial B_2} = \text{Identity}.$$



Greenleaf-Lassas-Uhlmann (2003)

$$\left.\begin{array}{l} \gamma = I : \text{Identity matrix in } B_2, \\ \tilde{\gamma} = \begin{cases} F_*\gamma & \text{in } B_2 \backslash \overline{B_1} \\ \text{arbitrary } \gamma_a & \text{in } B_1 \end{cases} \end{array}\right\} \Rightarrow \boxed{\Lambda_{\tilde{\gamma}} = \Lambda_\gamma}.$$

- Removable singularity argument (only works for Laplacian).

Outline

Maxwell's equations and Impedance map

- Maxwell's equations of electromagnetic fields $E(x)$ and $H(x)$ (as 1-forms):

$$\nabla \times E = i\omega\mu H \quad \nabla \times H = -i\omega\varepsilon E + J \quad \text{in } \Omega$$

  with permittivity $\varepsilon(x)$ and permeability $\mu(x)$ (as linear operators that map 1-forms to 2-forms).
- For **regular (Nonsingular)** medium: ε and μ are pos. def. sym. and bounded from below, $(E, H) \in H(\text{curl}) \times H(\text{curl})$.
- **Impedance map**:

$$\Lambda_{\mu,\varepsilon} : \nu \times E|_{\partial\Omega} \mapsto \nu \times H|_{\partial\Omega}.$$

Transformation invariance

Let $F : \Omega \to \Omega$ be a diffeomorphism. Denote $y = F(x)$.

- Pullback fields by F^{-1}:

$$\tilde{E}(y) = (F^{-1})^* E := (DF^T)^{-1} E \circ F^{-1}(y) \quad \text{similarly define } \tilde{H}(y),$$

$$\tilde{J}(y) = (F^{-1})^* J := [\det (DF)]^{-1} DF\, J \circ F^{-1}$$

- Push-forward of medium by F:

$$\tilde{\mu}(y) = F_* \mu := \frac{1}{\det (DF(x))} DF(x)\, \mu(x)\, DF(x)^T \Big|_{x=F^{-1}(y)} \quad \text{similarly define } \tilde{\varepsilon}(y).$$

- Then

$$\boxed{\nabla \times \tilde{E} = i\omega\tilde{\mu}\tilde{H}, \quad \nabla \times \tilde{H} = -i\omega\tilde{\varepsilon}\tilde{E} + \tilde{J}} \quad \text{in } \Omega$$

- Moreover, if $F|_{\partial\Omega} = \text{Identity}$, we have

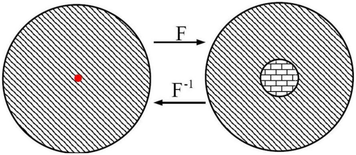
$$\boxed{\Lambda_{\tilde{\mu},\tilde{\varepsilon}} = \Lambda_{\mu,\varepsilon}}$$

Electromagnetic ideal cloaking design in B_2

$$F : B_2 \backslash \{0\} \to B_2 \backslash \overline{B_1}$$

$$F(x) = \left(1 + \frac{|x|}{2}\right) \frac{x}{|x|}.$$

$$F|_{\partial B_2} = \text{Identity}.$$



Greenleaf-Kurylev-Lassas-Uhlmann (2007)

$$\left.\begin{array}{l} (\mu, \varepsilon) = (I, I) \ \text{ in } B_2 \\ (\tilde{\mu}, \tilde{\varepsilon}) = \begin{cases} (F_* I, F_* I) & \text{in } B_2 \backslash \overline{B_1} \\ (\mu_0, \varepsilon_0) \text{ arbitrary} & \text{in } B_1 \end{cases} \end{array}\right\} \Rightarrow \boxed{\Lambda_{\tilde{\mu}, \tilde{\varepsilon}} = \Lambda_{I,I}}.$$

Singular cloaking medium

- Cloaking device medium:

$$\tilde{\mu} = \tilde{\varepsilon} = F_* I = \boxed{2\frac{(|x|-1)^2}{|x|^2}\Pi(x)} + 2(I - \Pi(x))$$

where $\Pi(x) = \hat{x}\hat{x}^T = xx^T/|x|^2$ is the projection along the radial direction.

- Degenerate singularity at $|x| = 1^+$!
- Notions of solutions for singular systems:
  - Physically meaningful
  - Forms with degenerate singular weighted energy may not be distributions
  - In what sense the equations are satisfied?
  - The cloaked region is considered isolated from the cloaking device or not?

Finite energy solutions $(\tilde{E}, \tilde{H})$ (Greenleaf-Kurylev-Lassas-Uhlmann).

Blow-up-a-small-ball regularization scheme

- **Nonsingular** transformation that blows up $B_\rho (0 < \rho < 1)$ to B_1 and fixes the boundary ∂B_2.

$$F_\rho(y) := \begin{cases} \left(\frac{2(1-\rho)}{2-\rho} + \frac{|y|}{2-\rho}\right)\frac{y}{|y|}, & \rho < |y| < 2, \\ \frac{y}{\rho}, & |y| < \rho. \end{cases}$$

- Regularized cloaking for Scalar optics and Acoustics (Helmholtz equations): Kohn-Onofrei-Vogelius-Weinstein
  - Scattering estimates of small inhomogeneity. Analysis as $\rho \to 0^+$ (Nguyen; Nguyen-Vogelius)
- Truncation-based regularization scheme for Helmholtz equations in 3D: Greenleaf-Kurylev-Lassas-Uhlmann

EM approximate cloaking

- Construct **regular** EM anisotropic material

$$(\tilde{\mu}_\rho, \tilde{\varepsilon}_\rho) := \begin{cases} ((F_\rho)_* I, (F_\rho)_* I), & 1 < |x| < 2, \\ (\mu_0, \varepsilon_0), & |x| < 1. \end{cases}$$

- $$(F_\rho)_* I = \frac{\left((2-\rho)|x| - 2 + 2\rho\right)^2}{(2-\rho)|x|^2} \Pi(x) + (2-\rho)(I - \Pi(x))$$

- Well-posedness: well-defined $H(\text{curl})$ solutions satisfying transmission problems in both physical space (cloaking device + cloaked region) and virtual space (pullback of physical space).
- Is $\Lambda_{\tilde{\mu}_\rho, \tilde{\varepsilon}_\rho} \approx \Lambda_{I,I}$? Yes and No.

EM waves in physical and virtual space

Physical space:

$$\left.\begin{aligned}\nabla\times\tilde{E}_\rho^+ &= i\omega\tilde{\mu}_\rho\tilde{H}_\rho^+,\\ \nabla\times\tilde{H}_\rho^+ &= -i\omega\tilde{\varepsilon}_\rho\tilde{E}_\rho^+ + \tilde{J},\end{aligned}\right\}\ \text{in } B_2\backslash\overline{B_1},$$

$$\left.\begin{aligned}\nabla\times\tilde{E}_\rho^- &= i\omega\mu_0\tilde{H}_\rho^-,\\ \nabla\times\tilde{H}_\rho^- &= -i\omega\varepsilon_0\tilde{E}_\rho^- + \tilde{J},\end{aligned}\right\}\ \text{in } B_1,$$

$$\nu\times\tilde{E}_\rho^+|_{\partial B_1^+} = \nu\times\tilde{E}_\rho^-|_{\partial B_1^-},$$

$$\nu\times\tilde{H}_\rho^+|_{\partial B_1^+} = \nu\times\tilde{H}_\rho^-|_{\partial B_1^-},$$

$$\nu\times\tilde{E}_\rho^+|_{\partial B_2} = f.$$

Virtual space:

$$\left.\begin{aligned}\nabla\times E_\rho^+ &= i\omega H_\rho^+,\\ \nabla\times H_\rho^+ &= -i\omega E_\rho^+ + J,\end{aligned}\right\}\ \text{in } B_2\backslash\overline{B_\rho},$$

$$\left.\begin{aligned}\nabla\times E_\rho^- &= i\omega((F_\rho^{-1})_*\mu_0)H_\rho^-,\\ \nabla\times H_\rho^- &= -i\omega((F_\rho^{-1})_*\varepsilon_0)E_\rho^- + J,\end{aligned}\right\}\ \text{in } B_\rho,$$

$$\nu\times E_\rho^+|_{\partial B_\rho^+} = \nu\times E_\rho^-|_{\partial B_\rho^-},$$

$$\nu\times H_\rho^+|_{\partial B_\rho^+} = \nu\times H_\rho^-|_{\partial B_\rho^-},$$

$$\nu\times E_\rho^+|_{\partial B_2} = f.$$

Cloaking a passive medium: $\tilde{J} = 0$

Assume μ_0 and ε_0 are positive constants.

- Spherical expansion:

$$\tilde{E}_\rho^- = \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \alpha_n^m M_{n,k\omega}^m + \beta_n^m \nabla \times M_{n,k\omega}^m,$$

$$E_\rho^+ = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_n^m N_{n,\omega}^m + d_n^m \nabla \times N_{n,\omega}^m + \gamma_n^m M_{n,\omega}^m + \eta_n^m \nabla \times M_{n,\omega}^m.$$

- Systems of linear equations for coefficients with coefficients matrix A
- Convergence order as $\rho \to 0$:

$$\gamma_n^m = O(1),\ \eta_n^m = O(1);\ c_n^m = O(\rho^{2n+1}),\ d_n^m = O(\rho^{2n+1});$$
$$\alpha_n^m = O(\rho^{n+1}),\ \beta_n^m = O(\rho^{n+1}).$$

Cloaking a passive medium: $\tilde{J} = 0$

Theorem 1 (Liu-Z)

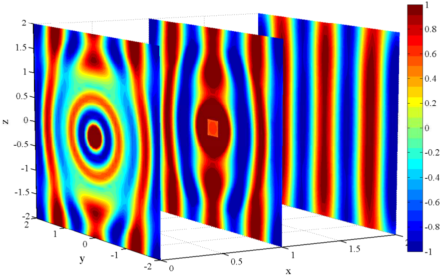
Suppose ω is not an eigenvalue of the transmission problems. Then we have

$$\|\Lambda_{\tilde{\mu},\tilde{\varepsilon}} - \Lambda_{I,I}\| = \mathcal{O}(\rho^3) \qquad \text{as } \rho \to 0^+.$$

boundary condition from the interior

$$\nu \times \tilde{E}_\rho^-|_{\partial B_1^-} \to 0, \quad \nu \times \tilde{H}_\rho^-|_{\partial B_1^-} \to 0.$$

Demonstration (passive)



$\mathrm{Re}(\tilde{E}_\rho)_1$ (sliced at $x = 0, 1, 2)$, $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, $\rho = 1/6$.

ρ	0.1	0.05	0.01	0.005	0.002	0.001
$Er(\rho)$	0.1810	0.0139	$8.42e-05$	$1.02e-06$	$6.42e-07$	$7.97e-08$
$r(\rho)$		3.703	3.173	3.044	3.020	3.009

Boundary errors and convergence order when $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$.

Cloaking an active medium: $\tilde{J} \neq 0$ supported in B_1

Given an internal point current $\tilde{J} = \sum_{|\alpha|<K} (\partial_x^\alpha \delta_0(x)) \mathbf{v}_\alpha$ at the origin,

- Spherical expansion:

$$\tilde{E}_\rho^- = \varepsilon_0^{-1/2} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \alpha_n^m M_{n,k\omega}^m + \beta_n^m \nabla \times M_{n,k\omega}^m + p_n^m N_{n,k\omega}^m + q_n^m \nabla \times N_{n,k\omega}^m,$$

$$E_\rho^+ = \sum_{n=1}^{\infty} \sum_{m=-n}^{n} c_n^m N_{n,\omega}^m + d_n^m \nabla \times N_{n,\omega}^m + \gamma_n^m M_{n,\omega}^m + \eta_n^m \nabla \times M_{n,\omega}^m.$$

- Convergence order as $\rho \to 0$:

$$\gamma_n^m = O(1),\ \eta_n^m = O(1);\ c_n^m = O(\rho^{n+1}),\ d_n^m = O(\rho^{n+1});$$
$$\alpha_n^m = O(1),\ \beta_n^m = O(1).$$

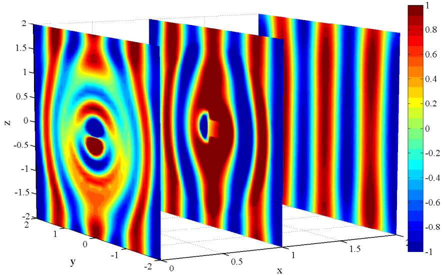
Cloaking an active medium: $\tilde{J} \neq 0$ supported in B_1

Theorem 2 (Liu-Z)

With an internal point current $\tilde{J} = \sum_{|\alpha|<K} (\partial_x^\alpha \delta_0(x))\mathbf{v}_\alpha$ at the origin, if ω is not an eigenvalue of the transmission problems, we have

$$\|\Lambda_{\tilde{\mu},\tilde{\varepsilon}} - \Lambda_{I,I}\| = \mathcal{O}(\rho^2) \qquad \text{as} \;\; \rho \to 0^+.$$

Demonstration (active)



$\mathrm{Re}(\tilde{E}_\rho)_1$ (sliced at $x = 0, 1, 2$), $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, $\rho = 1/12$, with a point source.

ρ	0.1	0.05	0.01	0.005	0.002	0.001
$Er(\rho)$	1.9787	0.3509	0.0114	0.0028	$4.41e-04$	$1.10e-04$
$r(\rho)$		2.495	2.129	2.031	2.013	2.006

Boundary errors and convergence order $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, with a point source.

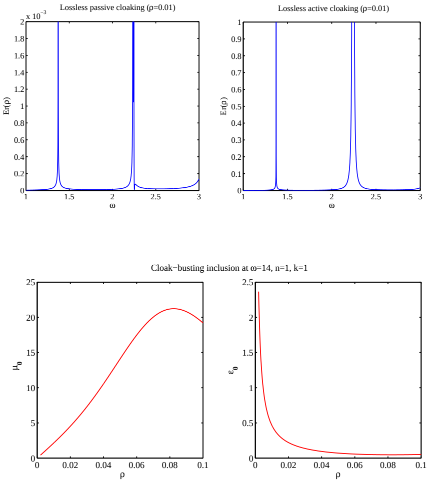
Resonance and Cloak-busting inclusion

- For a fixed cloaking scheme, i.e., fixed $\rho > 0$, there exists some frequency ω and cloaked medium (μ_0, ε_0) such that the transmission problems are **NOT well-posed**. Therefore, the boundary measurement $\Lambda_{\tilde{\mu},\tilde{\varepsilon}}$ is significantly different from $\Lambda_{I,I}$. For example, when $(\omega, \mu_0, \varepsilon_0)$ satisfies for some n

$$\mu_0 \frac{j_n(k\omega)}{\mathcal{J}_n(k\omega)} = \rho \frac{j_n(\omega\rho)h_n^{(1)}(2\omega) - h_n^{(1)}(\omega\rho)j_n(2\omega)}{\mathcal{J}_n(\omega\rho)h_n^{(1)}(2\omega) - \mathcal{H}_n(\omega\rho)j_n(2\omega)}$$

  where $k = (\mu_0\varepsilon_0)^{1/2}$.
- ω is the resonant frequency;
- (μ_0, ε_0) is called cloak-busting inclusion.

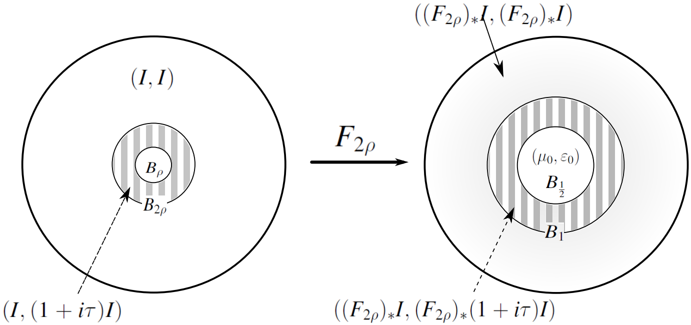
Demonstration (Resonance and Cloak-busting inclusion)



Boundary error $Er(\rho)$ for mode $n = 1$, when $\rho = 0.01$ and $\mu_0 = \varepsilon_0 = 2$, against frequency $\omega \in [1, 3]$ (Left: passive; Right: active).

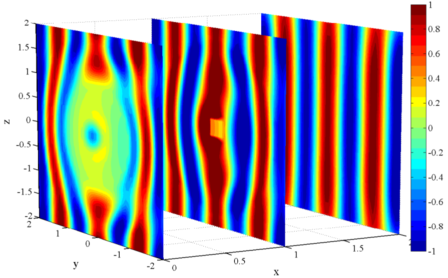
Cloak-busting inclusion medium μ_0 (left) and ε_0 (right) against $\rho \in (0, 0.1)$ at $\omega = 14$ for mode $n = 1$. Notice the singular behavior of the coefficients near the cloaking surface in resonant modes!

Remedy: Lossy layer



- $F_{2\rho}$ blows up $B_{2\rho}$ to B_1, τ is the damping parameter (conductivity).
- Spherical expansion of EM fields in three layers.
- Lossy regularization for the Helmholtz equations (Kohn-Onofrei-Vogelius-Weinstein, Kohn-Nguyen).
- Remedies by SS lining, SH lining (FSH lining) (Liu).

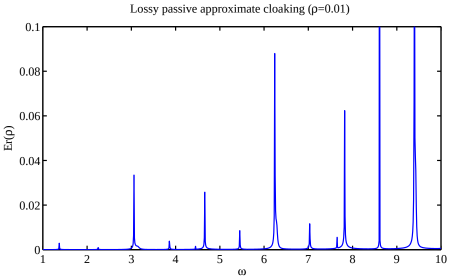
Demonstration (lossy)



$\mathrm{Re}(\tilde{E}_\rho)_1$ when $\omega = 5$, $\varepsilon_0 = \mu_0 = 2$, $\rho = 1/6$ and $\tau = 3$ for the lossy cloaking of a passive medium.

- The boundary convergence order is the same as the lossless cloaking (3 for passive, 2 for active).

Demonstration (lossy)



Boundary error $Er(\rho)$ for mode $n = 1$ when $\rho = 0.01$ of lossy approximate cloaking (passive), against frequency $\omega \in [1, 10]$.

- Resonant frequencies disappear.
- Observation: At some frequencies $Er(\rho)$ is relatively large. Are such frequencies the *poles* or *transmission eigenvalues* in the complex plane?

Outline

Ideal cloaking for the Helmholtz equations

- The Helmholtz equation for acoustics or scalar optics, with a source term p, inverse of the anisotropic mass density $\sigma = (\sigma^{jk})$ and the bulk modulus λ

$$\lambda \nabla \cdot \sigma \nabla u + \omega^2 u = p \quad \text{in } \Omega.$$

- **Dirichlet to Neumann map:** $\Lambda_{\sigma,\lambda} : u|_{\partial\Omega} \mapsto \nu \cdot \sigma \nabla u|_{\partial\Omega}$.
- **Cloaking medium** in the whole space $\mathbb{R}^2$:

$$(\tilde{\sigma}, \tilde{\lambda}) = \begin{cases} (I, 1) & |x| > 2 \\ (F_* I, F_* 1) & 1 < |x| \le 2 \\ (\sigma_a, \lambda_a) & |x| \le 1 \end{cases}$$

where $F_* \lambda(x) := [\det(DF)\lambda] \circ F^{-1}(x)$.

Singular cloaking medium

- Cloaking device medium: in $B_2 \backslash \overline{B_1}$

$$\tilde{\sigma} = F_* I = \boxed{\frac{|x|-1}{|x|}}\,\Pi(x) + \boxed{\frac{|x|}{|x|-1}}\,(I - \Pi(x))$$

$$\tilde{\lambda} = F_* 1 = \frac{|x|}{4(|x|-1)}$$

- Both degenerate and blow-up singularities at $|x| = 1^+$!

Truncation based regularization scheme

- **Regular** approximate inverse of mass tensor and bulk modulus with regularization parameter $1 < R < 2$

$$(\tilde{\sigma}_R, \tilde{\lambda}_R) = \begin{cases} (\tilde{\sigma}, \tilde{\lambda}) & |x| > R \\ (\sigma_a, \lambda_a) & |x| \leq R \end{cases}$$

- Case without internal source $p = 0$ in B_R:
Greenleaf-Kurylev-Lassas-Uhlmann

Cloaking a homogeneous medium with an internal source

Suppose (σ_a, λ_a) is constant. Set $\kappa^2 = (\sigma_a\lambda_a)^{-1}$ and $\rho = F^{-1}(R)$

- **Physical space:**

$$(\tilde{\lambda}\nabla\cdot\tilde{\sigma}\nabla + \omega^2)u_R^+ = p, \quad \text{in } B_2\backslash\overline{B_R}$$

$$\boxed{(\Delta + \kappa^2\omega^2)u_R^- = \kappa^2 p \quad \text{in } B_R}$$

- **Virtual space:** $v_R^+ = u_R^+ \circ F$,

$$\boxed{(\Delta + \omega^2)v_R^+ = p\circ F \quad \text{in } B_2\backslash\overline{B_\rho}}$$

- Transmission conditions and boundary conditions:

$$\boxed{\begin{array}{l} v_R^+|_{\partial B_\rho^+} = u_R^-|_{\partial B_R^-}, \quad \rho\partial_r v_R^+|_{\partial B_\rho^+} = \kappa R\partial_r u_R^-|_{\partial B_R^-}, \\ v_R^+|_{\partial B_2} = f. \end{array}}$$

Cloaking a homogeneous medium with an internal source

Given $p \in C^\infty(\mathbb{R}^2)$ with $\mathrm{supp}(p) \subset B_{R_0}$ $(0 < R_0 < 1)$

- Spherical expansions:

$$u_R^-(\tilde{r},\theta) = \sum_{n=-\infty}^{\infty} (a_n J_{|n|}(\kappa\omega\tilde{r}) + p_n H_{|n|}^{(1)}(\kappa\omega\tilde{r}))e^{in\theta}, \quad \tilde{r} \in (R_0, R)$$

$$v_R^+(r,\theta) = \sum_{n=-\infty}^{\infty} (b_n J_{|n|}(\omega r) + c_n H_{|n|}^{(1)}(\omega r))e^{in\theta}, \quad r \in (\rho, 2)$$

- Linear system about a_n, b_n and c_n by the transmission conditions and boundary condition.

Resonance observations due to the internal source

Resonant frequency limit $\omega \Leftrightarrow$ **cloak-busting inclusion limit** $\kappa = (\sigma_0 \lambda_0)^{-1/2}$

$\Leftrightarrow \; |a_n|, |b_n|, |c_n| \to \infty$ as $R \to 1^+ (\rho \to 0^+)$ $(n \geq 1)$

$$\Leftrightarrow \; \boxed{\left[\omega\kappa^2 R (J_{|n|})'(\kappa\omega R) + |n| J_{|n|}(\kappa\omega R)\right]\Big|_{R=1} = 0}$$

$\Leftrightarrow V_{\pm n}(\tilde{r}, \theta) := J_{|n|}(\kappa\omega\tilde{r}) e^{\pm in\theta}$ are **eigenfunctions** of

$$\boxed{\begin{array}{l} (\Delta + \kappa^2\omega^2)V = 0 \quad \text{in } B_1, \\ [\kappa\tilde{r}\partial_{\tilde{r}} V + (-\partial_\theta^2)^{1/2} V]|_{\tilde{r}=1^+} = 0. \end{array}}$$

- A non-local boundary condition

Non-resonant result: non-local boundary conditions

Suppose ω and (σ_a, λ_a) satisfy

$$\begin{cases} [\omega\kappa^2 R (J_{|n|})'(\kappa\omega R) + |n| J_{|n|}(\kappa\omega R)]\big|_{R=1} \neq 0, \\ J_{|n|}(2\omega) \neq 0, \end{cases} \qquad \text{for } n \in \mathbb{Z}.$$

Lassas-Z

As $R \to 1^+$, u_R (the solution in the physical space) converges uniformly in compact subsets of $B_2 \backslash \partial B_1$ to the limit u_1 satisfying

$$(\Delta + \kappa^2\omega^2)u_1 = \kappa^2 p \quad \text{in } B_1,$$

$$[\kappa\partial_{\tilde{r}} u_1 + (-\partial_\theta^2)^{1/2} u_1]\big|_{\partial B_1} = 0.$$

Conclusions

- Electromagnetic non-singular approximate cloaking
  - Convergence orders are 3 for passive cloaking and 2 for active cloaking for both lossless and lossy cloaking;
- 2D acoustic cloaking: A non-local boundary condition is obtained as $R \to 1^+$.

❦❦❦ Thank you! ❦❦❦