Geometric Analysis on Euclidean and Homogeneous Spaces

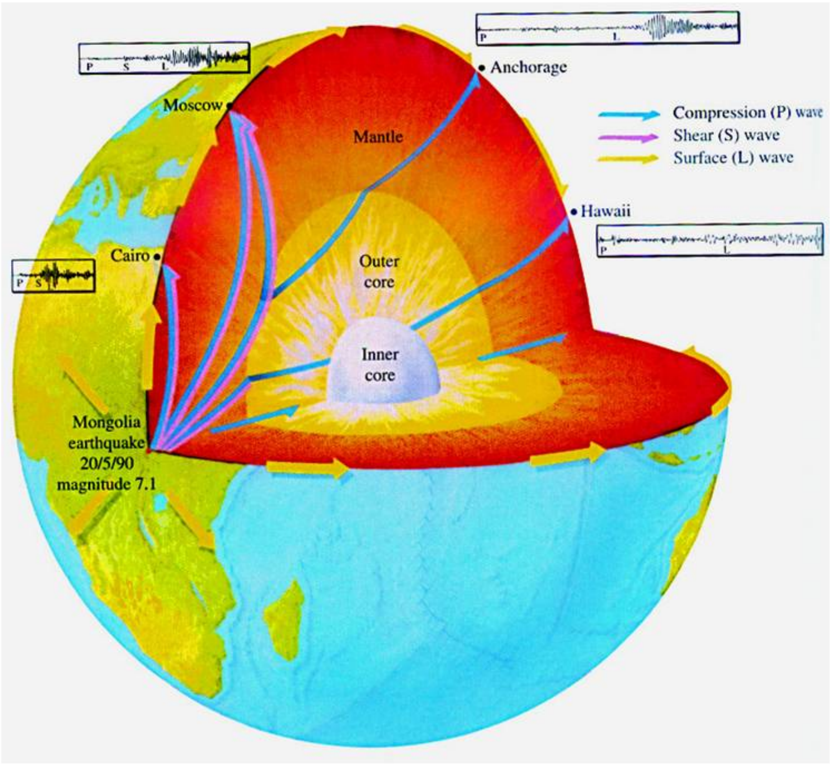
Travel Time Tomography and Tensor Tomography

Gunther Uhlmann

UC Irvine & University of Washington

Tufts University, January, 2012

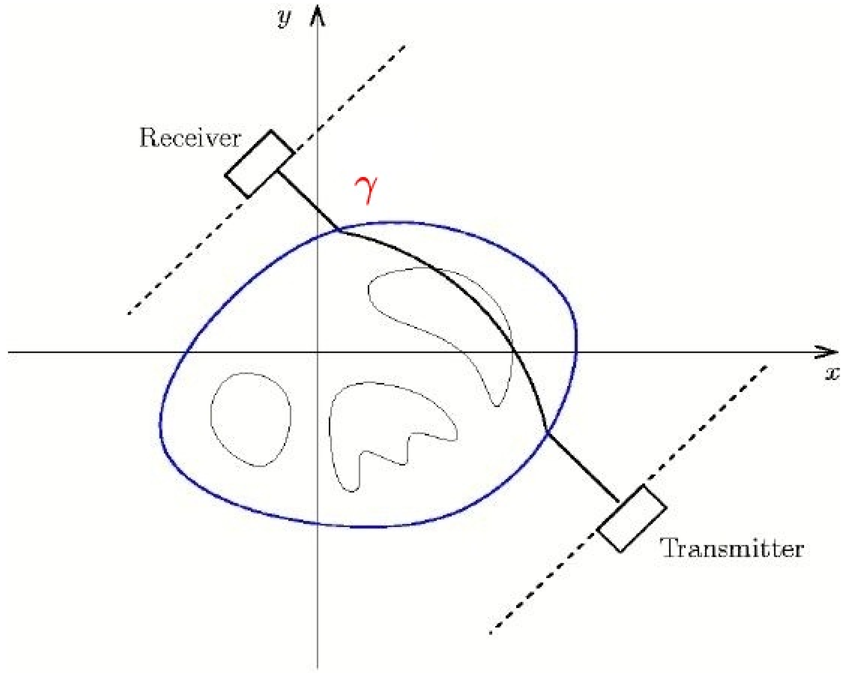
Global Seismology



Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

Human Body Seismology

ULTRASOUND TRANSMISSION TOMOGRAPHY(UTT)

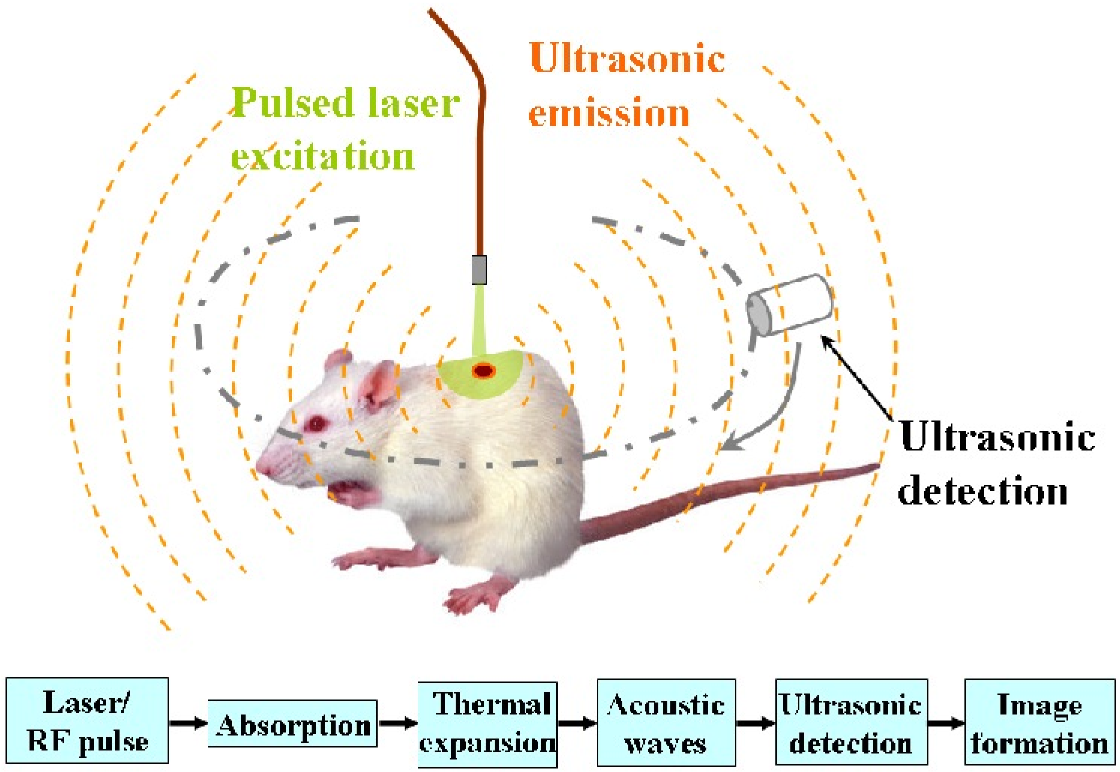


$$T = \int_{\gamma} \frac{1}{c(x)}\, ds$$ = Travel Time (Time of Flight).

TechniScan

(Loading TechniScan.mp4)

Thermoacoustic Tomography



Wikipedia

Mathematical Model

First Step: in PAT and TAT is to reconstruct $H(x)$ from $u(x,t)|_{\partial\Omega\times(0,T)}$, where u solves

$$\begin{aligned}(\partial_t^2 - c^2(x)\Delta)u &= 0 \quad \text{on } \mathbb{R}^n \times \mathbb{R}^+ \\ u|_{t=0} &= \beta H(x) \\ \partial_t u|_{t=0} &= 0\end{aligned}$$

Second Step: in PAT and TAT is to reconstruct the optical or electrical properties from $H(x)$ (internal measurements).

How to reconstruct $c(x)$?

Proposal: To use **UTT** (Y. Xin-L. V. Wang, Phys. Med. Biol. 51 (2006) 6437–6448).

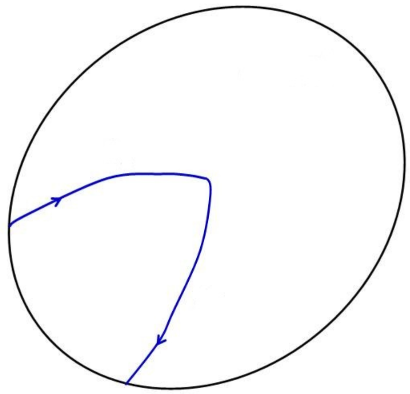
THIRD MOTIVATION

OCEAN ACOUSTIC TOMOGRAPHY

Ocean Acoustic Tomography

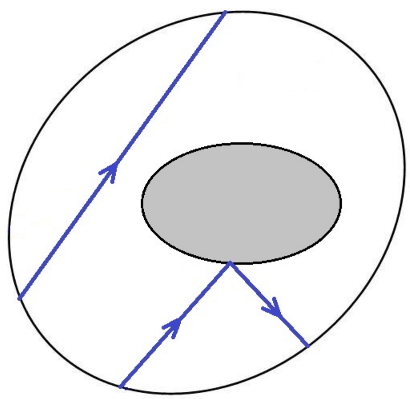
Ocean Acoustic Tomography is a tool with which we can study average temperatures over large regions of the ocean. By measuring the time it takes sound to travel between known source and receiver locations, we can determine the soundspeed. Changes in soundspeed can then be related to changes in temperature.

REFLECTION TOMOGRAPHY



Scattering

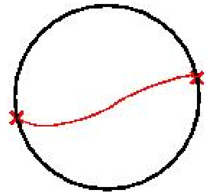
Points in medium



Obstacle

TRAVELTIME TOMOGRAPHY (Transmission)

Motivation:Determine inner structure of Earth by measuring travel times of seismic waves



Herglotz, Wiechert-Zoeppritz (1905)

Sound speed $c(r)$, $r = |x|$

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

Reconstruction method of $c(r)$ from lengths of geodesics

$$ds^2 = \frac{1}{c^2(r)} dx^2$$

More generally $ds^2 = \frac{1}{c^2(x)} dx^2$

Velocity $v(x, \xi) = c(x), \quad |\xi| = 1$ (isotropic)

Anisotropic case

$$ds^2 = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j$$

$g = (g_{ij})$ is a positive definite symmetric matrix

Velocity $v(x, \xi) = \sqrt{\sum_{i,j=1}^{n} g^{ij}(x) \xi_i \xi_j}, \quad |\xi| = 1$

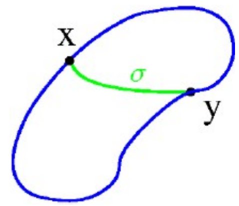
$$g^{ij} = (g_{ij})^{-1}$$

The information is encoded in the boundary distance function

More general set-up

(M, g) a Riemannian manifold with boundary

(compact) $g = (g_{ij})$



$x, y \in \partial M$

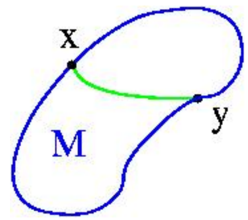
$$d_g(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

$L(\sigma) =$ length of curve σ

$$L(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

Inverse problem

Determine g knowing $d_g(x,y)$ $x, y \in \partial M$



$dg \Rightarrow g$?

(Boundary rigidity problem)

Answer NO $\quad \psi : M \to M$ diffeomorphism

$$\psi\Big|_{\partial M} = \text{Identity}$$

$$d_{\psi^* g} = d_g$$

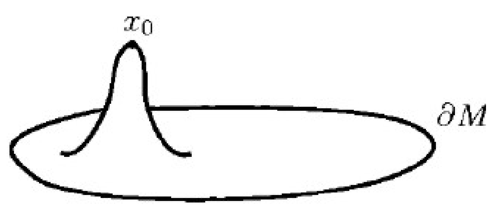
$$\psi^* g = \left(D\psi \circ g \circ (D\psi)^T\right) \circ \psi$$

$$L_g(\sigma) = \int_0^1 \sqrt{\sum_{i,j=1}^n g_{ij}(\sigma(t)) \frac{d\sigma_i}{dt} \frac{d\sigma_j}{dt}} dt$$

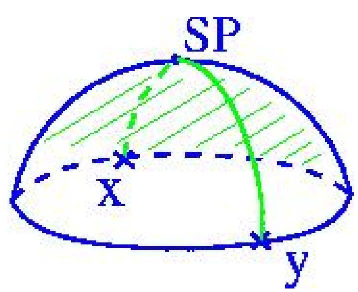
$$\tilde{\sigma} = \psi \circ \sigma \quad L_{\psi^* g}(\tilde{\sigma}) = L_g(\sigma)$$

$$\boxed{d_{\psi^* g} = d_g}$$

Only obstruction to determining g from d_g ? No



$$d_g(x_0, \partial M) > \sup_{x,y \in \partial M} d_g(x, y)$$



Can change metric near SP

Acoustic Shadow Zone

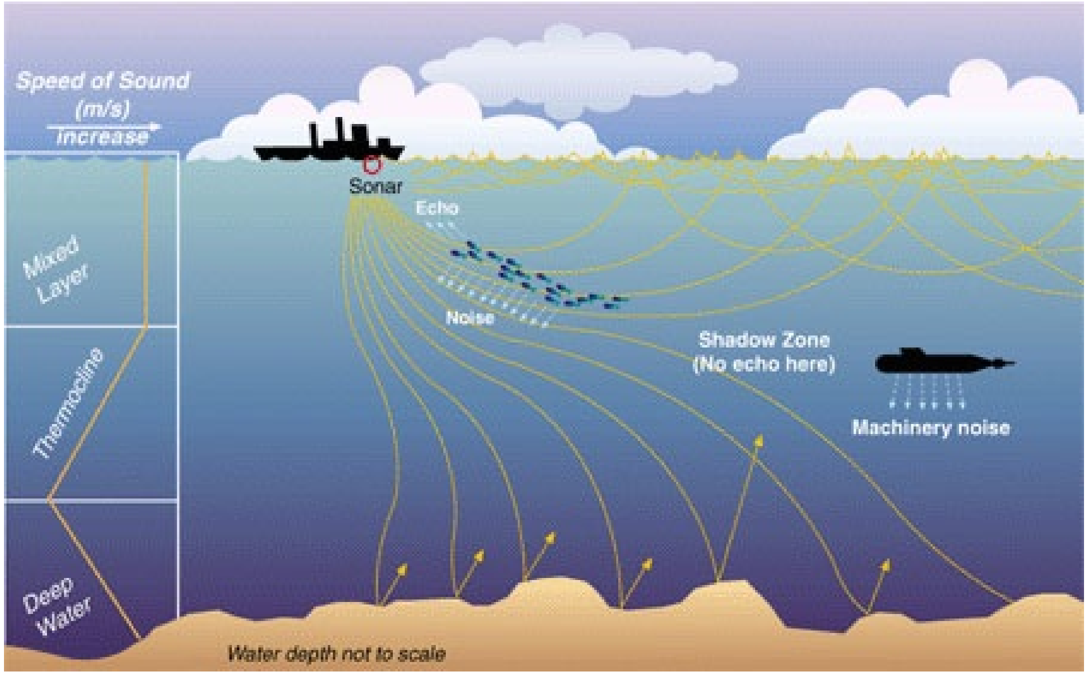


Figure: Beyond Discovery: Sounding Out the Ocean's Secrets by Victoria Kaharl

Def (M, g) is boundary rigid if $(M, \widetilde{g})$ satisfies $d_{\widetilde{g}} = d_g$. Then $\exists \psi : M \to M$ diffeomorphism, $\psi\Big|_{\partial M} =$ Identity, so that

$$\boxed{\widetilde{g} = \psi^* g}$$

Need an a-priori condition for (M, g) to be boundary rigid.

One such condition is that (M, g) is simple

DEF (M,g) is simple if given two points $x, y \in \partial M$, $\exists !$ geodesic joining x and y and ∂M is strictly convex

CONJECTURE

(M,g) is simple then (M,g) is boundary rigid ,that is $\boxed{d_g}$ determines $\boxed{g}$ up to the natural obstruction. $(d_{\psi^* g} = d_g)$

(Conjecture posed by R. Michel, 1981)

Results (M, g) simple

- R. Michel (1981) Compact subdomains of $\mathbb{R}^2$ or $\mathbb{H}^2$ or the open round hemisphere

- Gromov (1983) Compact subdomains of $\mathbb{R}^n$

- Besson-Courtois-Gallot (1995) Compact subdomains of negatively curved symmetric spaces

(All examples above have constant curvature)

- $\left\{\begin{array}{l}\text{Lassas-Sharafutdinov-U (2003)} \\ \text{Burago-Ivanov (2010)}\end{array}\right\}$ $\boxed{dg = dg_0}$, g_0 close to Euclidean

- $\boxed{n = 2}$ Otal and Croke (1990) $\boxed{K_g < 0}$

$\boxed{n = 2}$

THEOREM(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

Theorem ($n \geq 3$) (Stefanov-U, 2005)

There exists a generic set $\widetilde{\mathcal{L}} \subset C^k(M) \times C^k(M)$ such that

$$(g_1, g_2) \in \widetilde{\mathcal{L}},\ g_i \text{ simple},\ i = 1, 2,\ d_{g_1} = d_{g_2}$$

$$\Longrightarrow \exists \psi :\ M \to M \text{ diffeomorphism,}$$

$$\psi\Big|_{\partial M} = \text{Identity, so that } \boxed{g_1 = \psi^* g_2}.$$

Remark

If M is an open set of $\mathbb{R}^n$, $\widetilde{\mathcal{L}}$ contains all pairs of simple and real-analytic metrics in $C^k(M)$.

Theorem ($n \geq 3$) (Stefanov-U, 2005)

(M, g_i) simple $i = 1, 2$, g_i close to $g_0 \in \mathcal{L}$ where $\mathcal{L}$ is a generic set of simple metrics in $C^k(M)$. Then

$$d_{g_1} = d_{g_2} \Rightarrow \exists \psi : \; M \to M \text{ diffeomorphism,}$$

$$\psi\Big|_{\partial M} = \text{Identity, so that } \boxed{g_1 = \psi^* g_2}$$

Remark

If M is an open set of $\mathbb{R}^n$, $\mathcal{L}$ contains all simple and real-analytic metrics in $C^k(M)$.

Geodesics in Phase Space

$g = \big(g_{ij}(x)\big)$ symmetric, positive definite

Hamiltonian is given by

$$H_g(x,\xi) = \frac{1}{2}\Big(\sum_{i,j=1}^{n} g^{ij}(x)\xi_i\xi_j - 1\Big) \qquad g^{-1} = \big(g^{ij}(x)\big)$$

$X_g(s,X^0) = \big(x_g(s,X^0), \xi_g(s,X^0)\big)$ be bicharacteristics,

sol. of $$\frac{dx}{ds} = \frac{\partial H_g}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_g}{\partial x}$$

$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_g^{n-1}(x^0)$

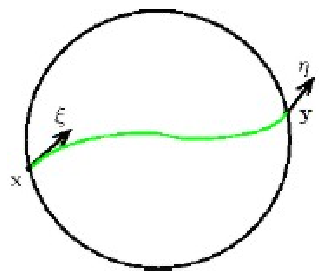
$$\mathcal{S}_g^{n-1}(x) = \Big\{\xi \in \mathbb{R}^n;\ H_g(x,\xi) = 0\Big\}.$$

Geodesics Projections in x: $x(s)$.

Scattering Relation

d_g only measures first arrival times of waves.

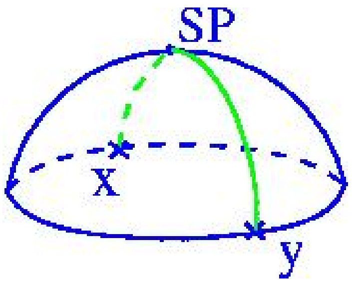
We need to look at behavior of all geodesics



$$\|\xi\|_g = \|\eta\|_g = 1$$

$\alpha_g(x, \xi) = (y, \eta)$, α_g is SCATTERING RELATION

If we know direction and point of entrance of geodesic then we know its direction and point of exit (plus travel time).



Scattering relation follows all geodesics.

Conjecture Assume (M,g) non-trapping. Then α_g determines g up to natural obstruction.

(Pestov-U, 2005) $n = 2$ Connection between α_g and Λ_g (Dirichlet-to-Neumann map)

(M, g) simple then $d_g \Leftrightarrow \alpha_g$

Theorem (Vargo, 2009)

(M_i, g_i), $i = 1, 2$, compact Riemannian real-analytic manifolds with boundary satisfying a mild condition. Assume

$$\boxed{\alpha_{g_1} = \alpha_{g_2}}$$

Then $\exists \psi : M \to M$ diffeomorphism, such that

$$\boxed{\psi^* g_1 = g_2}$$

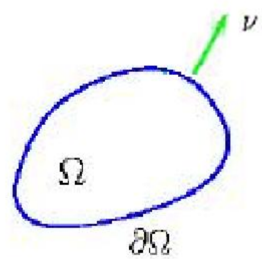
Dirichlet-to-Neumann Map (Lee–U, 1989)

(M,g) compact Riemannian manifold with boundary.

Δ_g Laplace-Beltrami operator $g=(g_{ij})$ pos. def. symmetric matrix

$$\Delta_g u = \frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(\sqrt{\det g}\, g^{ij} \frac{\partial u}{\partial x_j}\right) \quad (g^{ij}) = (g_{ij})^{-1}$$

$$\begin{aligned} \Delta_g u &= 0 \text{ on } M \\ u\big|_{\partial M} &= f \end{aligned}$$



Conductivity:
$\gamma^{ij} = \sqrt{\det g}\, g^{ij}$

$$\Lambda_g(f) = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\,\Big|_{\partial M}$$

$\nu = (\nu^1, \cdots, \nu^n)$ unit-outer normal

$$\Delta_g u = 0$$
$$u\big|_{\partial M} = f$$

$$\boxed{\Lambda_g(f) = \frac{\partial u}{\partial \nu_g} = \sum_{i,j=1}^{n} \nu^j g^{ij} \frac{\partial u}{\partial x_i} \sqrt{\det g}\bigg|_{\partial M}}$$

current flux at ∂M

Inverse-problem (EIT)

Can we recover $\boxed{g}$ from $\boxed{\Lambda_g}$?

$\Lambda_g =$ Dirichlet-to-Neumann map or voltage to current map

Theorem $(n = 2)$(Lassas-U, 2001)

(M, g_i), $i = 1, 2$, connected Riemannian manifold with boundary. Assume

$$\Lambda_{g_1} = \Lambda_{g_2}$$

Then $\exists \psi : M \to M$ diffeomorphism, $\psi\big|_{\partial M} = \text{Identity}$, and $\beta > 0, \beta\big|_{\partial M} = 1$ so that

$$g_1 = \beta \psi^* g_2$$

In fact, one can determine topology of M as well.

$\boxed{n = 2}$

THEOREM(Pestov-U, 2005)

Two dimensional Riemannian manifolds with boundary which are simple are boundary rigid ($d_g \Rightarrow g$ up to natural obstruction)

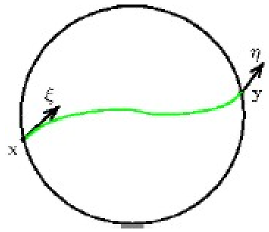
CONNECTION BETWEEN BOUNDARY RIGIDITY AND DIRICHLET-TO-NEUMANN MAP

THEOREM $(n = 2)$ (Pestov-U, 2005)

If we know d_g then we can determine Λ_g if (M, g) simple.

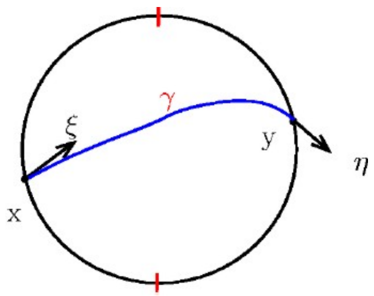
IN FACT (M, g) simple $n = 2$

$$d_g \Rightarrow \alpha_g \Rightarrow \Lambda_g$$



$$\alpha_g(x, \xi) = (y, \eta)$$

CONNECTION BETWEEN SCATTERING RELATION AND DIRICHLET-TO-NEUMANN MAP$(n = 2)$



$$\alpha_g(x, \xi) = (y, \eta)$$

$\boxed{d_g}$ determines $\boxed{\Lambda_g}$ if geodesic X-ray transform injective

$$\boxed{If(x,\xi) = \int_\gamma f} \qquad If = 0 \implies f = 0$$

Now $\Lambda_g \overset{L-U}{\Longrightarrow} \beta \psi^* g, \ \beta > 0$

If I is injective, we can also recover β.

Dirichlet-to-Neumann map	Boundary distance function
$\Lambda_g(f)(x) = \int_{\partial M} \lambda_g(x,y) f(y) dS_y$ λ_g depends on 2n-2 variables	$d_g(x,y), \quad x, y \in \partial M$ $d_g(x,y)$ dep. on 2n-2 variables
$\Delta_g u = 0, \quad u\big\vert_{\partial M} = f$ $\Lambda_g \Longleftrightarrow Q_g$ $Q_g(f) = \sum \int_M g^{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dx$ $= \inf_{v\vert_{\partial M} = f} \int_M g^{ij} \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} dx$	$d_g(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L_g(\sigma)$ $L_g(\sigma) = \int_0^1 \sqrt{g_{ij}(\sigma(t)) \frac{\partial \sigma_i}{\partial t} \frac{\partial \sigma_j}{\partial t}} dt$

Dirichlet-to-Neumann map

$$\Delta_g u = 0$$
$$u\Big|_{\partial M} = f$$
$$\Lambda_g(f) = \frac{\partial u}{\partial \nu_g}$$

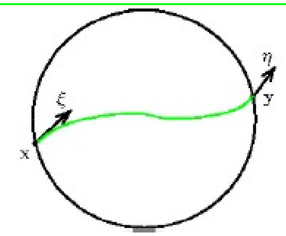
$$\Big\{(f, \Lambda_g(f))\Big\} \subseteq L^2(\partial M) \times L^2(\partial M)$$

is Lagrangian manifold

g=e=Euclidean

$$\langle (f_1, g_1), (f_2, g_2) \rangle$$
$$= \int_{\partial M} (g_1 f_2 - f_1 g_2) dS$$

(Scattering relation)



$$H_g(x, \xi) = \frac{1}{2}\Big(\sum g^{ij}\xi_i\xi_j - 1\Big)$$
$$\frac{dx_g}{ds} = +\frac{\partial H_g}{\partial \xi}$$
$$\frac{d\xi_g}{ds} = -\frac{\partial H_g}{\partial x}$$
$$x_g(0) = x, \quad \xi_g(0) = \xi, \quad \|\xi\|_g = 1$$

we know $(x_g(T), \xi_g(T))$

$$\alpha_g(x, \xi) = (y, \eta)$$

$\Big\{(x, \xi), \alpha_g(x, \xi)\Big\}$ projected to $T^*(\partial M) \times T^*(\partial M)$ is Lagrangian manifold

TENSOR TOMOGRAPHY

Linearized Boundary Rigidity Problem

Recover a tensor (f_{ij}) from the geodesic X-ray transform

$$I_g f(\gamma) = \int_\gamma f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j dt$$

known for all maximal geodesics γ on M.

$$f = f^s + dv, \quad v|_{\partial M} = 0$$

and $\delta f^s = 0$, δ=divergence. $I_g(dv) = 0$.

Linearized Problem To recover f^s from $I_g f$.

Stefanov-U, 2005 If we solve this we solve the boundary rigidity problem locally (near a metric).

TENSOR TOMOGRAPHY

$$I_g f(\gamma) = \int_\gamma f_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j dt,$$
$$f = (f_{ij}) = f^s + dv, \quad v|_{\partial M} = 0$$

Recover f^s from $I_g f$.

<u>**Theorem**</u> ($n = 2$) (Sharafutdinov 2007, Paternain-Salo-U 2011) (M, g) simple. Then I_g is injective on solenoidal vector fields.

<u>Remark</u> Also stability estimates are valid (Stefanov-U 2005). This implies stability for non-linear problem (boundary rigidity).

<u>**Theorem**</u> ($n \geq 3$) (Stefanov-U 2005) (M, g) simple, g real-analytic. Then I_g is injective on solenoidal vector fields.

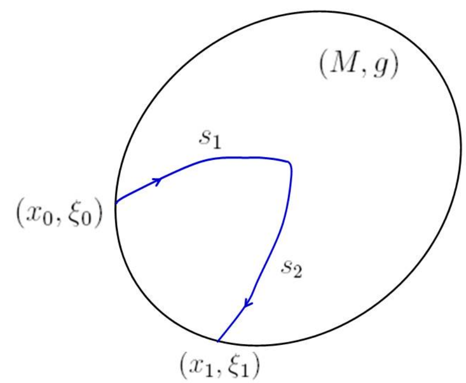
<u>Remark</u> This implies solution of boundary rigidity near real-analytic metrics and also stability.

REFLECTION TRAVELTIME TOMOGRAPHY

Broken Scattering Relation

(M, g): manifold with boundary with Riemannian metric $\boxed{g}$

$$\boxed{\begin{array}{c}((x_0, \xi_0), (x_1, \xi_1), t) \in \mathcal{B} \\ t = s_1 + s_2\end{array}}$$



Theorem (Kurylev-Lassas-U 2010)

$n \geq 3$. Then ∂M and the broken scattering relation $\mathcal{B}$ determines (M, g) uniquely.

Identity (Stefanov-U, 1998)

$$\int_0^T \frac{\partial X_{g_2}}{\partial X^0}\left(T-s, X_{g_1}(s,X^0)\right)(V_{g_1}-V_{g_2})\Big|_{X_{g_1}(s,X^0)} dS$$
$$= X_{g_1}(T,X^0) - X_{g_2}(T,X^0)$$

$$V_{g_j} := \left(\frac{\partial H_{g_j}}{\partial \xi}, -\frac{\partial H_{g_j}}{\partial x}\right)$$ the Hamiltonian vector field.

Particular case:

$$(g_k) = \frac{1}{c_k^2}\left(\delta_{ij}\right), \qquad k = 1, 2$$

$$V_{g_k} = \left(c_k^2 \xi,\ -\frac{1}{2}\nabla(c_k^2)|\xi|^2\right)$$

Linear in c_k^2!

Reconstruction

$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\left(T-s, X_{g_2}(s,X^0)\right) \times \left((c_1^2-c_2^2)\xi,\, -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\right)\Big|_{X_{g_2}(s,X^0)} dS = \underbrace{X_{g_1}(T,X^0)}_{\text{data}} - X_{g_2}(T,X^0)$$

Inversion of geodesic X-ray transform.

Consider X^*X for inversion.

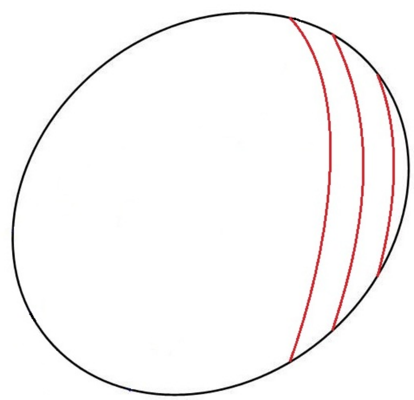
Numerical Method

(Chung-Qian-Zhao-U, IP 2011)

$$\int_0^T \frac{\partial X_{g_1}}{\partial X^0}\Big(T-s, X_{g_2}(s,X^0)\Big) \times \Big((c_1^2-c_2^2)\xi, -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{g_2}(s,X^0)} dS = X_{g_1}(T,X^0) - X_{g_2}(T,X^0)$$

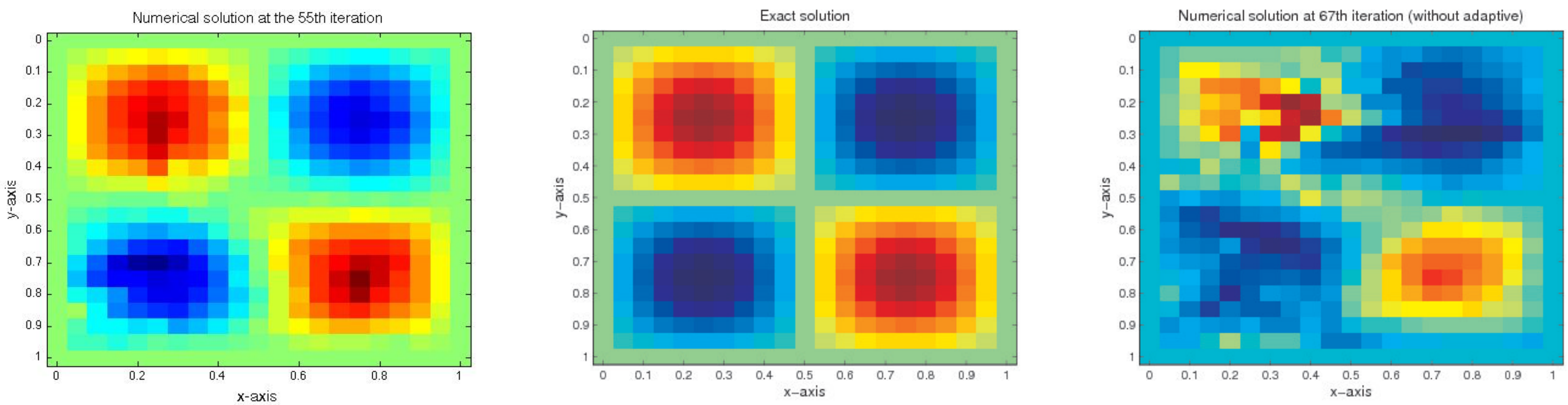
Adaptive method

Start near $\partial\Omega$ with $c_2 = 1$ and iterate.



Numerical examples

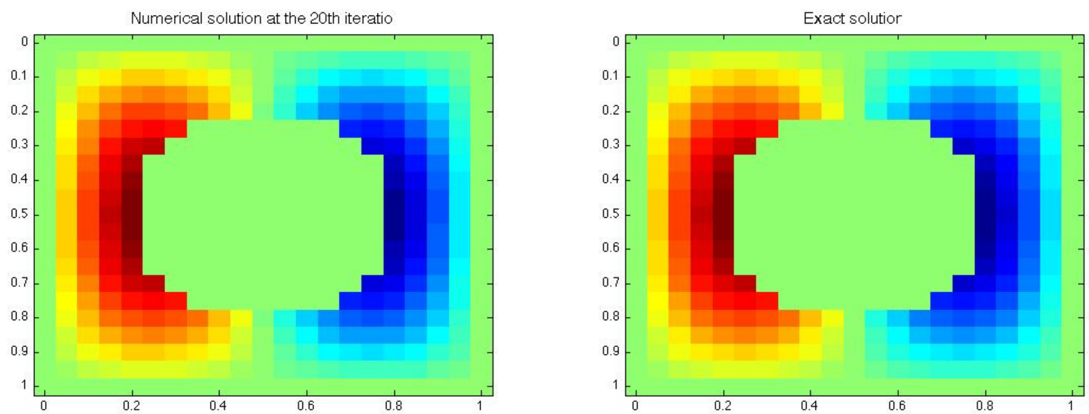
Example 1: An example with no broken geodesics, $c(x,y) = 1 + 0.3\sin(2\pi x)\sin(2\pi y)$, $c_0 = 0.8$.



Left: Numerical solution (using adaptive) at the 55-th iteration. Middle: Exact solution. Right: Numerical solution (without adaptive) at the 67-th iteration.

Example 2: A known circular obstacle enclosed by a square domain. Geodesic either does not hit the inclusion or hits the inclusion (broken) once.

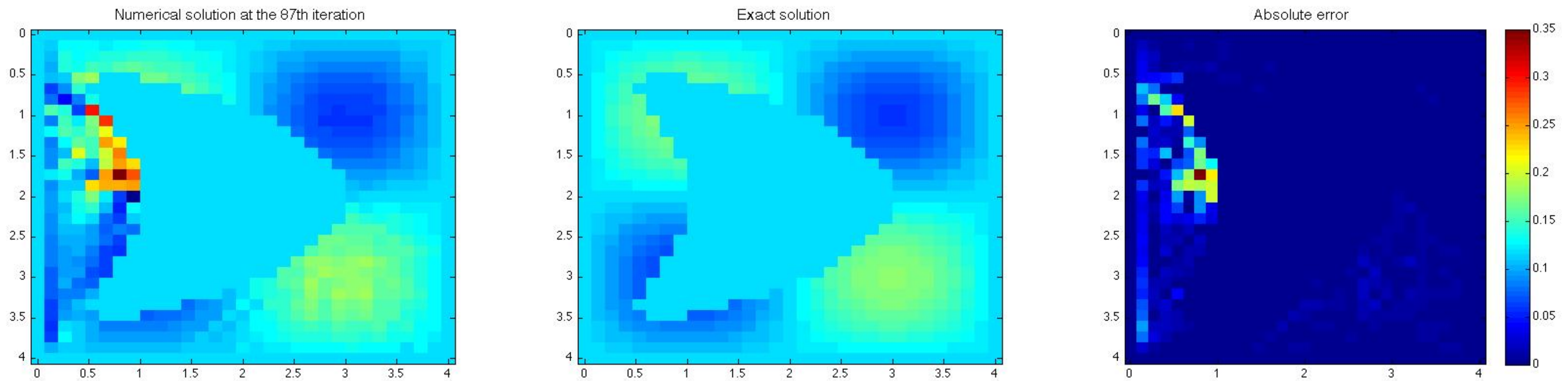
$c(x, y) = 1 + 0.2\sin(2\pi x)\sin(\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 20-th iteration. The relative error is 0.094%. Right: Exact solution.

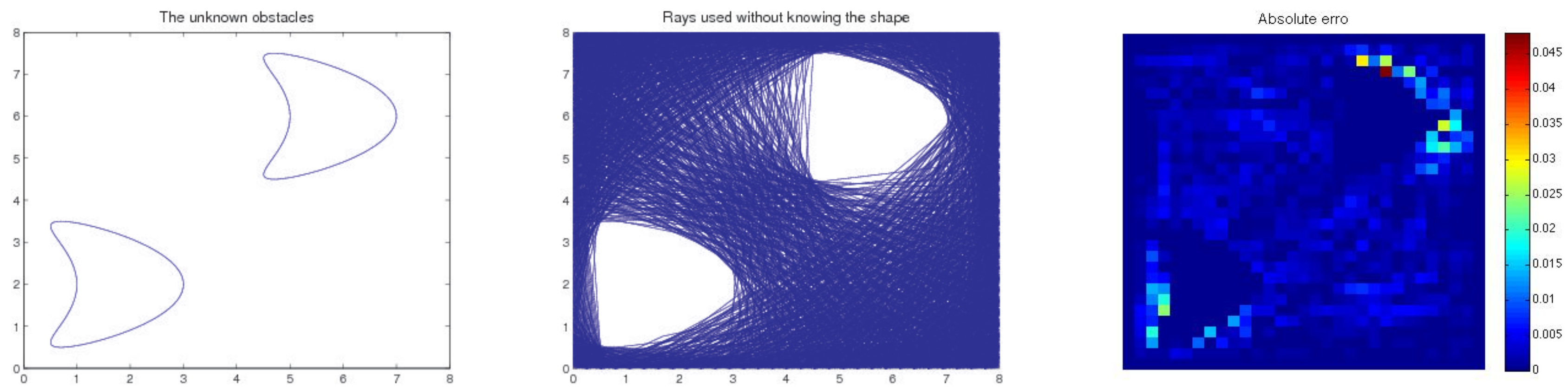
Example 3: A concave obstacle (known).

$c(x, y) = 1 + 0.1 \sin(0.5\pi x) \sin(0.5\pi y)$, $c_0 = 0.8$.



Left: Numerical solution at the 117-th iteration. The relative error is 2.8%. Middle: Exact solution. Right: Absolute error.

Example 4: Unknown obstacles and medium.

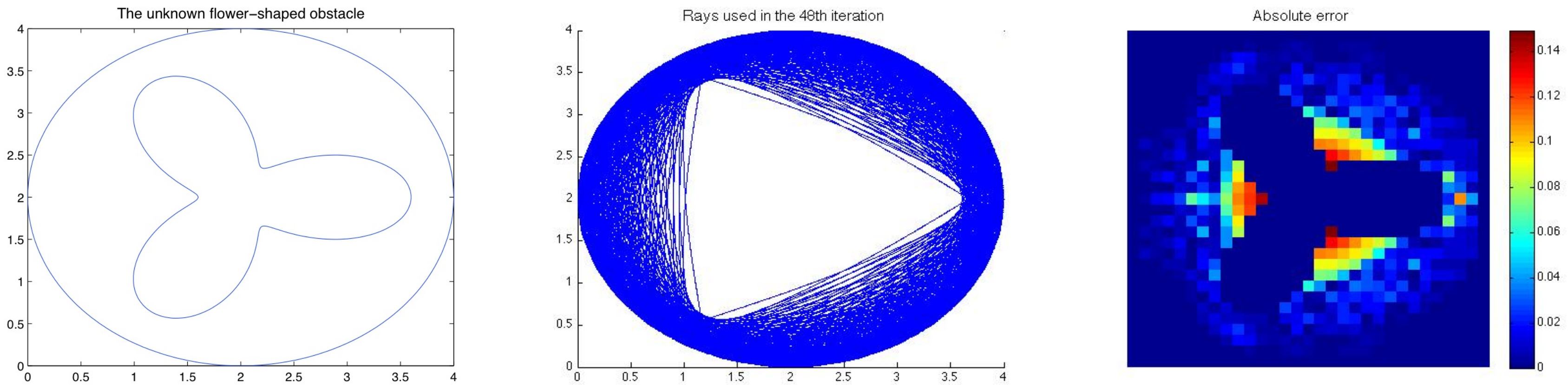


Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

Example 4: Unknown obstacles and medium (continues).

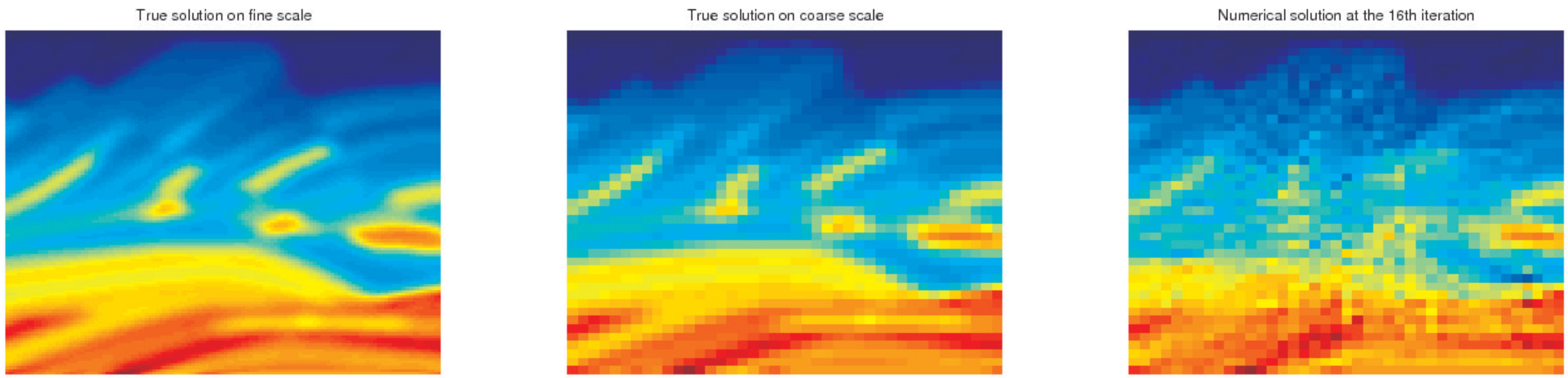
$r = 1 + 0.6\cos(3\theta)$ with $r = \sqrt{(x-2)^2 + (y-2)^2}$.

$$c(r) = 1 + 0.2\sin r$$



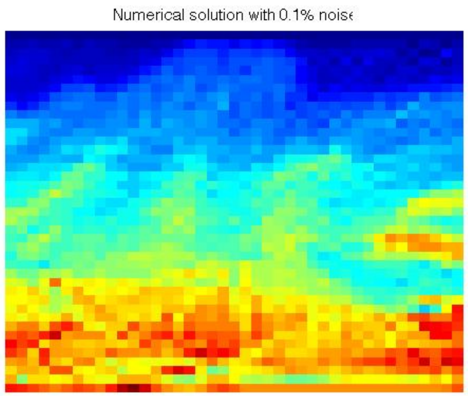
Left: The two unknown obstacles. Middle: Ray coverage of the unknown obstacle. Right: Absolute error.

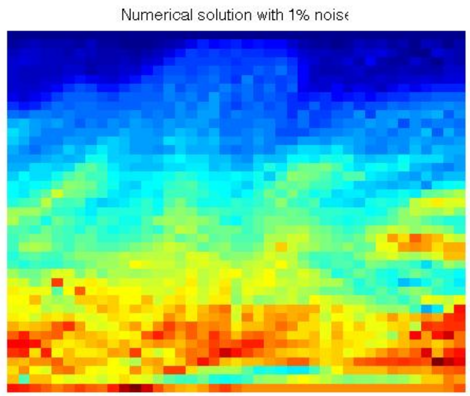
Example 5: The Marmousi model.



Left: The exact solution on fine grid. Middle: The exact solution projected on a coarse grid. Right: The numerical solution at the 16-th iteration. The relative error is 2.24%.

Example 5: The Marmousi model (with noise).





Left: The numerical solution with 0.1% noise. The relative error is 4.16%. Right: The numerical solution with 1% noise. The relative error is 5.53%.

Dirichlet-to-Neumann map		Boundary distance function (Scattering relation)
$n=2$ (M,g) simple Λ_g	$\Longleftarrow$	$\begin{cases} d_g(x,y) \\ \alpha_g(x,\xi)=(y,\eta) \end{cases}$
$n=2$ (M,g) Λ_g	$\overset{?}{\Longleftarrow}$	α_g
$n=3$ Λ_g $\Lambda_g=\Lambda_e$, e=Euclidean $g=\psi^*e$?	$\overset{?}{\Longrightarrow}$	d_g or α_g $d_g=d_e \Rightarrow g=\psi^*e$ (Gromov)