Inversion Formulas for Spherical Means in Constant Curvature Spaces

Yu. Antipov, R. Estrada, B. Rubin (LSU)

Workshop on Geometric Analysis
on Euclidean and Homogeneous Spaces

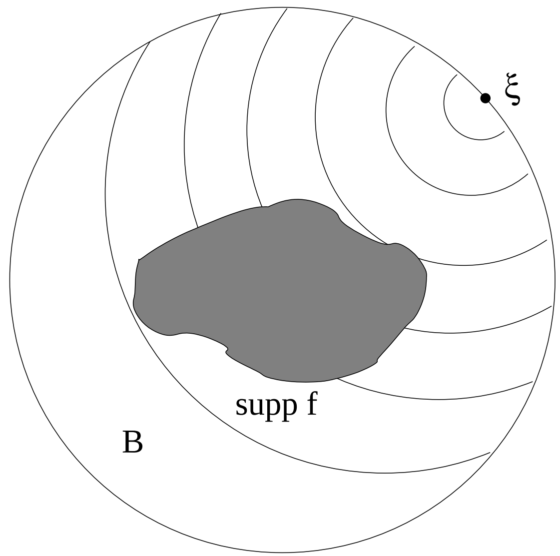
Tufts University, January 8-9, 2012

Setting of the Problem.

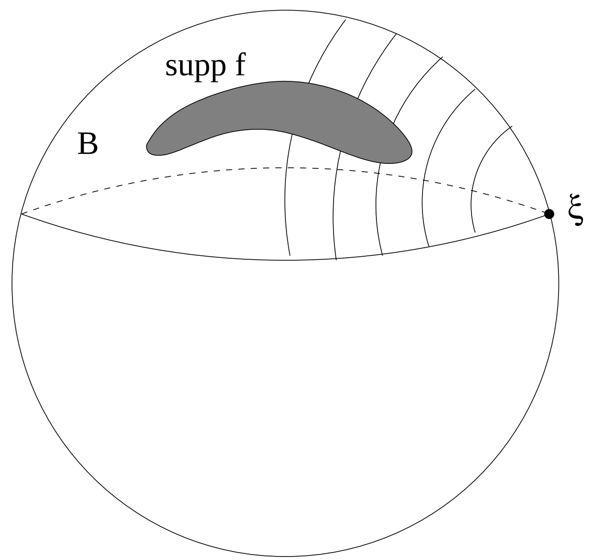
Let B be an open ball of radius R in an n-dimensional constant curvature space $X \in \{\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n\}$. Reconstruct a function f supported in B, if the spherical means of f are known over all geodesic spheres centered on the boundary ∂B.

Setting of the Problem.

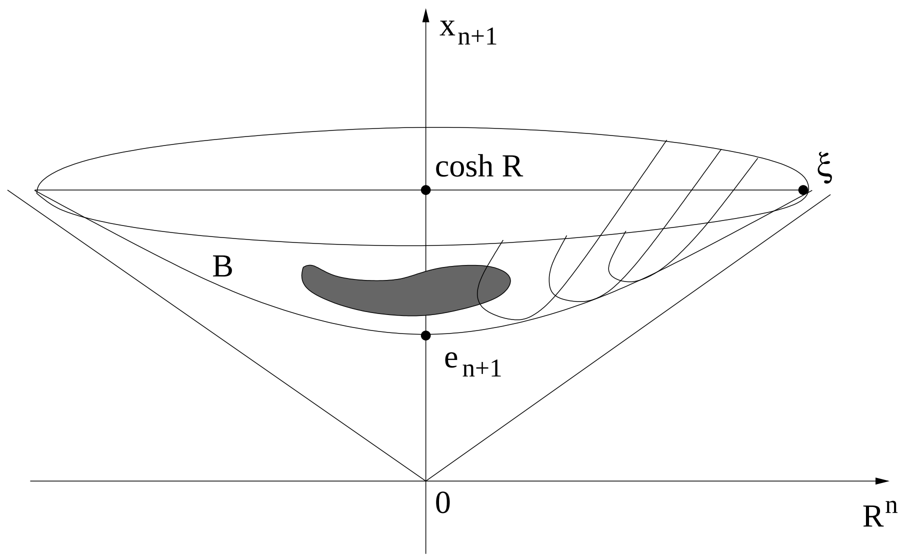
Let B be an open ball of radius R in an n-dimensional constant curvature space $X \in \{\mathbb{R}^n, \mathbb{S}^n, \mathbb{H}^n\}$. Reconstruct a function f supported in B, if the spherical means of f are known over all geodesic spheres centered on the boundary ∂B.



$$X = \mathbb{R}^n$$



$$X = \mathbb{S}^n$$



$$X = \mathbb{H}^n$$

References ($X = \mathbb{R}^n$):

- D. Finch, S. K. Patch, Rakesh, 2004 (n odd);
- V. Palamodov, 2004 (n odd);
- D. Finch, M. Haltmeier, Rakesh, 2007 (n even);
- L. Kunyansky, 2007 (any n);
- B.R., 2008 (n odd);
- M. Agranovsky, P. Kuchment, E.T. Quinto, W. Madych, Linh Nguyen, and their collaborators.

Main Results. The case $X = \mathbb{R}^n$

Notation:

$$B = \{x \in \mathbb{R}^n : |x| < R\},$$

$$(Mf)(\xi, t) = \int_{S^{n-1}} f(\xi - t\sigma)\, d\sigma, \qquad (\xi, t) \in \partial B \times \mathbb{R}_+,$$

$$f \in C^\infty(\mathbb{R}^n), \qquad \text{supp}(f) \subset B,$$

$$D = \frac{d}{d(t^2)} = \frac{1}{2t}\frac{d}{dt}, \qquad \Delta = \sum_{k=1}^{n} \frac{\partial^2}{\partial x_k^2}.$$

Theorem.

If $n = 3, 5, \ldots,$ then

$$f(x) = d_{n,1}\, \Delta \int\limits_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi, t)]\Big|_{t=|x-\xi|} d\xi.$$

Theorem.

If $n = 3, 5, \ldots$, then

$$f(x) = d_{n,1}\, \Delta \int\limits_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|}\, d\xi.$$

If $n = 2, 4, 6, \ldots$, then

$$f(x) = d_{n,2}\, \Delta \int\limits_{\partial B} d\xi \int\limits_{0}^{2R} t\, D^{n-2}[t^{n-2}(Mf)(\xi,t)]\, \log\left|t^2 - |x-\xi|^2\right| dt;$$

$$d_{n,1} = \frac{(-1)^{(n-1)/2}\,\pi^{1-n/2}}{4R\,\Gamma(n/2)}, \qquad d_{n,2} = \frac{(-1)^{n/2-1}\,\pi^{-n/2}}{2R\,(n/2-1)!}.$$

The case $X = \mathbb{S}^n$

Notation:

$$B_\theta = \{x \in \mathbb{S}^n : x \cdot e_{n+1} > \cos\theta\}, \quad e_{n+1} = (0, \ldots, 0, 1), \; \theta \in (0, \pi/2];$$

$$(Mf)(x,t) = \frac{(1-t^2)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{x \cdot y = t} f(y)\, d\sigma(y), \quad x \in \mathbb{S}^n, \; t \in (-1, 1),$$

$$f \in C^\infty(\mathbb{S}^n), \qquad \mathrm{supp}(f) \subset B_\theta;$$

For $x \in B_\theta$ we write

$$x = (x', \sqrt{1 - |x'|^2}), \qquad x' = (x_1, \ldots, x_n, 0).$$

Theorem. Let $f \in C^\infty(\mathbb{S}^n)$, $\operatorname{supp} f \subset B_\theta$. We denote

$$f_0(x) = -\int_{\partial B_\theta} (d/dt)^{n-3}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\Big|_{t=\xi\cdot x}\, d\xi$$

if $n = 3, 5, \dots,$ and

$$f_0(x) = \frac{1}{\pi}\int_{\partial B_\theta} d\xi \int_{\cos 2\theta}^{1} (d/dt)^{n-2}[(Mf)(\xi,t)\,(1-t^2)^{n/2-1}]\,\log|t-\xi\cdot x|\,dt$$

if $n = 2, 4, \dots$. Then f can be reconstructed by the formula

$$f(x) = \frac{d_n\, x_{n+1}}{\sin\theta}\,\Delta_{x'} f_0(x', \sqrt{1-|x'|^2}), \qquad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $x = (x', \sqrt{1-|x'|^2})$.

The case $X = \mathbb{H}^n$

Notation: $x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1},\ n \geq 2$;

$[x, y] = -x_1 y_1 - \cdots - x_n y_n + x_{n+1} y_{n+1}$, dist$(x, y) = \cosh^{-1}[x, y]$;

$$B = \{x \in \mathbb{H}^n : \text{dist}(x, e_{n+1}) < R\} = \{x \in \mathbb{H}^n : x_{n+1} < \cosh R\};$$

$$(Mf)(x,t) = \frac{(t^2-1)^{(1-n)/2}}{\sigma_{n-1}} \int\limits_{[x,y]=t} f(y)\, d\sigma(y), \qquad x \in \mathbb{H}^n,\ t > 1,$$

$$f \in C^\infty(\mathbb{H}^n), \qquad \text{supp}(f) \subset B;$$

For $x \in B$ we write

$$x = (x', \sqrt{1 + |x'|^2}), \qquad x' = (x_1, \dots, x_n, 0).$$

Theorem. Let $f \in C^\infty(\mathbb{H}^n)$, $\operatorname{supp} f \subset B$. We denote

$$f_0(x) = -\int_{\partial B} (d/dt)^{n-3}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\Big|_{t=[\xi,x]}\,d\xi$$

if $n = 3, 5, \ldots,$ and

$$f_0(x) = \frac{1}{\pi}\int_{\partial B} d\xi \int_1^{\cosh 2R} (d/dt)^{n-2}[(Mf)(\xi,t)\,(t^2-1)^{n/2-1}]\,\log|t - [\xi,x]|\,dt$$

if $n = 2, 4, \ldots$. Then f can be reconstructed by the formula

$$f(x) = \frac{d_n\, x_{n+1}}{|x|\sinh R}\,\Delta_{x'} f_0(x', \sqrt{1+|x'|^2}), \qquad d_n = \frac{(-1)^{[n/2-1]}}{2^{n-1}\pi^{n/2-1}\Gamma(n/2)},$$

where $x = (x', \sqrt{1+|x'|^2})$.

Proof of the Main Theorem (the case $X = \mathbb{R}^n, \ n > 2$)

The basic idea: analytic continuation (a.c.).

$$\begin{aligned}(N^\alpha f)(\xi, t) &= \int_B \frac{|t^2 - |y-\xi|^2|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)\,dy \\ &= \sigma_{n-1} \int\limits_0^{2R} \frac{|t^2 - r^2|^{\alpha-1}}{\Gamma(\alpha/2)} (Mf)(\xi, r)\, r^{n-1}\,dr,\end{aligned}$$

$$(\xi, t) \in \partial B \times \mathbb{R}_+, \qquad Re\,\alpha > 0, \qquad \sigma_{n-1} = |\mathbb{S}^{n-1}|.$$

Proof of the Main Theorem (the case $X = \mathbb{R}^n,\ n > 2$)

The basic idea: analytic continuation (a.c.).

$$\begin{aligned}(N^\alpha f)(\xi, t) &= \int_B \frac{|t^2 - |y-\xi|^2|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)\, dy \\ &= \sigma_{n-1} \int_0^{2R} \frac{|t^2 - r^2|^{\alpha-1}}{\Gamma(\alpha/2)} (Mf)(\xi, r)\, r^{n-1}\, dr,\end{aligned}$$

$$(\xi, t) \in \partial B \times \mathbb{R}_+, \qquad Re\, \alpha > 0, \qquad \sigma_{n-1} = |\mathbb{S}^{n-1}|.$$

The backprojection operator:

$$(PF)(x) = \frac{1}{|\partial B|} \int_{\partial B} F(\xi, |x-\xi|)\, d\xi, \qquad x \in B.$$

Lemma 1 (the basic). Let $n > 2$, $|h| < 1$. The integral

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int_{-1}^{1} |t-h|^{\alpha-1} (1-t^2)^{(n-3)/2} \, dt, \qquad Re\, \alpha > 0,$$

extends as an entire function of α . Moreover,

$$\underset{\alpha=3-n}{a.c.} \, g_\alpha(h) = \Gamma((n-1)/2).$$

Lemma 1 (the basic). Let $n > 2,\ |h| < 1$. The integral

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int_{-1}^{1} |t-h|^{\alpha-1} (1-t^2)^{(n-3)/2}\, dt, \qquad Re\,\alpha > 0,$$

extends as an entire function of α . Moreover,

$$\underset{\alpha=3-n}{a.c.}\, g_\alpha(h) = \Gamma((n-1)/2).$$

Proof. Use properties of hypergeometric functions.

Lemma 1 (the basic). Let $n > 2, \ |h| < 1$. The integral

$$g_\alpha(h) = \frac{1}{\Gamma(\alpha/2)} \int_{-1}^{1} |t-h|^{\alpha-1} (1-t^2)^{(n-3)/2} \, dt, \qquad Re\,\alpha > 0,$$

extends as an entire function of α . Moreover,

$$\underset{\alpha=3-n}{a.c.} \, g_\alpha(h) = \Gamma((n-1)/2).$$

Proof. Use properties of hypergeometric functions.

Lemma 2.

$$\underset{\alpha=3-n}{a.c.} (PN^\alpha f)(x) = \frac{\Gamma(n/2)}{\pi^{1/2} \, (2R)^{n-2}} \int_B \frac{f(y)}{|x-y|^{n-2}} \, dy.$$

Lemma 3. Let

$$D = \frac{1}{2t}\frac{d}{dt}, \qquad \delta_n = \frac{(-1)^{[n/2-1]}\,\Gamma((n-1)/2)}{(n-3)!}.$$

(i) If $n = 3, 5, \ldots$, then

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = \frac{\delta_n}{2R^{n-1}} \int_{\partial B} D^{n-3}[t^{n-2}(Mf)(\xi,t)]\Big|_{t=|x-\xi|} d\xi.$$

(ii) If $n = 4, 6, \ldots$, then

$$\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x) = -\frac{\delta_n}{\pi R^{n-1}} \int_{\partial B} d\xi$$

$$\times \int_0^{2R} t\, D^{n-2}[t^{n-2}(Mf)(\xi,t)]\, \log|t^2 - |x-\xi|^2|\, dt.$$

End of the proof.

Equate different forms of $\underset{\alpha=3-n}{a.c.}(PN^\alpha f)(x)$ in Lemmas 2 and 3, and then apply the Laplace operator to reconstruct f.

End of the proof.

Equate different forms of $\underset{\alpha=3-n}{a.c.}(PN^{\alpha}f)(x)$ in Lemmas 2 and 3, and then apply the Laplace operator to reconstruct f.

For $X = \mathbb{S}^n$ the proof relies on the same idea of analytic continuation of $PN^{\alpha}f$ with

$$(N^{\alpha}f)(\xi,t) = \int_{B_\theta} \frac{|\xi\cdot y - t|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)\,dy,$$

$$(\xi,t)\in\partial B_\theta\times(-1,1), \qquad Re\,\alpha>0,$$

$$(PF)(x) = \frac{1}{|\partial B_\theta|}\int_{\partial B_\theta} F(\xi,\xi\cdot x)\,d\xi, \qquad x\in B_\theta.$$

If $X = \mathbb{H}^n$ we set

$$(N^\alpha f)(\xi,t) = \int\limits_B \frac{|[\xi,y]-t|^{\alpha-1}}{\Gamma(\alpha/2)} f(y)\,dy,$$

$$(\xi,t) \in \partial B \times (1,\infty), \qquad Re\,\alpha > 0,$$

$$(PF)(x) = \frac{1}{|\partial B|} \int\limits_{\partial B} F(\xi,[\xi,x])\,d\sigma(\xi), \qquad x \in B,$$

and proceed as in the previous cases.

Thank you!