Band-limited Localized Parseval frames on Compact Homogeneous Manifolds

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Dedicated to S. Helgason on his 85-th Birthday

In the last decade, methods based on various kinds of spherical wavelet bases have found applications in virtually all areas where analysis of spherical data is required, including cosmology, weather prediction, and geodesy. In particular, the so-called needlets (=band-limited Parseval frames) have become an important tool for the analysis of Cosmic Microwave Background (CMB) temperature data.

The goal of the present paper is to construct band-limited and highly localized Parseval frames on general compact homogeneous manifolds. Our construction can be considered as an analogue of the well-known φ-transform on Euclidean spaces.

Frames were introduced in the 1950s by Duffin and Schaeffer to represent functions via over-complete sets. Frames arise naturally in wavelet analysis on Euclidean spaces when continuous wavelet transforms are discretized. They provide a redundancy that helps reduce the effect of noise in data, and they have been constructed, studied, and employed extensively in both theoretical and applied problems.

A set of vectors $\{\theta_v\}$ in a Hilbert space H is called a frame if there exist constants $A, B > 0$ such that for all $f \in H$

$$A\|f\|_2^2 \leq \sum_v | < f, \theta_v > |^2 \leq B\|f\|_2^2. \tag{1}$$

The largest A and smallest B are called lower and upper frame bounds. The set of scalars $\{< f, \theta_v >\}$ represents a set of measurements of a signal f. To synthesize signal f from this set of measurements one has to find another (dual) frame $\{\Theta_v\}$ and then a reconstruction formula is

$$f = \sum_v < f, \theta_v > \Theta_v. \tag{2}$$

Dual frame is not unique in general. Moreover it is not easy (and expensive) to find a dual frame.

However, if a frame is **tight**, which means that $A = B$, then synthesis can be performed by using the **original frame** $\{\theta_v\}$**.** If in particular $A = B = 1$ the frame is known as the **Parseval frame.**

Parseval frames are similar in many respects to orthonormal wavelet bases. For example, if in addition all vectors θ_v are unit vectors, then the frame is an **orthonormal basis.**

But the main feature of Parseval frames is that **decomposing and synthesizing a signal or image from known data are tasks carried out with the same set of functions**, the ones in the frame.

A property that makes one frame preferable to another is **simultaneous localization of the frame functions in both space and frequency.** Frames with this feature have been successfully developed in Euclidean spaces. One of them is the so-called φ-transform which was discovered in 90s by M. Frazier and B. Jawerth.

A *homogeneous* compact manifold $\mathbf{M}$ is a C^∞-compact manifold on which a compact Lie group G acts transitively. In this case $\mathbf{M}$ is necessary of the form G/K, where K is a closed subgroup of G. The notation $L_2(\mathbf{M})$, is used for the usual Hilbert spaces, where dx is an invariant measure.

If $\mathbf{g}$ is the Lie algebra of a compact Lie group G then it is a direct sum $\mathbf{g} = \mathbf{a} + [\mathbf{g}, \mathbf{g}]$, where $\mathbf{a}$ is the center of $\mathbf{g}$, and $[\mathbf{g}, \mathbf{g}]$ is a semi-simple algebra. Let Q be a positive-definite quadratic form on $\mathbf{g}$ which, on $[\mathbf{g}, \mathbf{g}]$, is opposite to the Killing form. Let $X_1, ..., X_d$ be a basis of $\mathbf{g}$, which is orthonormal with respect to Q. Since the form Q is $Ad(G)$-invariant, the operator

$$-X_1^2 - X_2^2 - \ ... - X_d^2,\ d = dim\ G$$

is a bi-invariant operator on G. This implies in particular that the corresponding operator on $L_p(\mathbf{M}),\ 1 \leq p \leq \infty,$

$$\mathcal{L} = -D_1^2 - D_2^2 - ... - D_d^2, \quad D_j = D_{X_j},\ d = dim\ G, \qquad (3)$$

commutes with all operators $D_j = D_{X_j}$.

This operator involved in most of the constructions and results of our research.
Operator $\mathcal{L}$, which is usually called the Laplace operator, is the image of the Casimir operator under differential of quazi-regular representation in $L_2(\mathbf{M})$.
The operator $\mathcal{L}$ is elliptic, positive definite and has a discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ......$ which goes to infinity.
Let $\{\varphi_j\}$ be a corresponding system of orthonormal eigenfunctions which forms a basis in $L_2(\mathbf{M})$.

Our strategy is the following.
Given a compact Riemannian manifold **M** with distance d we consider a sequence of what we call 2^{-j}-*lattices* M_j. By a 2^{-j} we mean a set of points

$$M_j = \{x_k^{(j)}\}_{k=1}^{\mathcal{N}_j}$$

which are "uniformly" distributed over **M** in the sense that there exist constants $a, b > 0$ which are independent on natural j such that for all $x_k^{(j)}, y_m^{(j)} \in M_j$

$$a2^{-j} \leq \max_k \min_{k \neq m} dist\,(x_k^{(j)}, y_m^{(j)}) \leq b2^{-j}.$$

With every node $x_k^{(j)}$ we associate a smooth function $\psi_k^{(j)}$ which is
1) "essentially" supported in a ball of radius 2^{-j} around $x_k^{(j)}$ and
2) is bandlimited in a sense it is a polynomial in eigenfunctions of $\mathcal{L}$.
Once again, the objective is to show that such construction can be performed in such way that the set of functions $\psi_k^{(j)}$ will form a Parseval frame i.e. one would have the equality

$$\|f\|_2^2 = \sum_{j,k} | < f, \psi_k^{(j)} > |^2$$

for any $f \in L_2(\mathbf{M})$.

Definition

The span of all eigenfunctions of $\mathcal{L}$ whose eigenvalues are not greater an $\omega > 0$ will be denoted as $\mathbf{E}_\omega(\mathcal{L})$.

For the operator $\mathcal{L}$ we were able to prove the following property which is of crucial importance for our construction.

Theorem

(Product property) If $\mathbf{M} = G/K$ is a compact homogeneous manifold and $\mathcal{L}$ is the same as above, then for any f and g belonging to $\mathbf{E}_\omega(\mathcal{L})$, their product fg belongs to $\mathbf{E}_{4d\omega}(\mathcal{L})$, where d is the dimension of the group G.

To introduce our second key result we need more preparations.

Lemma

For any compact Riemannian manifold $\mathbf{M}$ *there exists a natural number* $N_{\mathbf{M}}$*, such that for any sufficiently small* $\rho > 0$ *there exists a set of points* $\{y_\nu\}$ *such that:*

1. *the balls* $B(y_\nu, \rho/4)$ *are disjoint,*
2. *the balls* $B(y_\nu, \rho/2)$ *form a cover of* $\mathbf{M}$*,*
3. *the multiplicity of the cover by balls* $B(y_\nu, \rho)$ *is not greater than* $N_{\mathbf{M}}$*.*

Definition

Any set of points $M_\rho = \{y_\nu\}$ *which is as described in Lemma will be called a metric* ρ*-lattice.*

Theorem

(Cubature formula) There exists a positive constant a_0, such that if $\rho = a_0(\omega + 1)^{-1/2}$, then for any ρ-lattice M_ρ, there exist strictly positive coefficients $\lambda_{x_k} > 0,\ x_k \in M_\rho,$ for which the following equality holds for all functions in $\mathbf{E}_\omega(\mathbf{M})$:

$$\int_{\mathbf{M}} f dx = \sum_{x_k \in M_\rho} \lambda_{x_k} f(x_k). \tag{4}$$

Moreover, there exists constants $c_1,\ c_2$, such that the following inequalities hold:

$$c_1 \rho^n \leq \lambda_{x_k} \leq c_2 \rho^n,\ n = dim\, \mathbf{M}. \tag{5}$$

(Relevant results were obtained independently by H. Mhaskar, F. Filbir, F. Narcowich, J. Ward).

According to spectral theorem if F is a Schwartz function on the line, then there is a well defined operator $F(\mathcal{L})$ in the space $L_2(\mathbf{M})$ such that for any $f \in L_2(\mathbf{M})$ one has

$$\left(F(\mathcal{L})f\right)(x) = \int_{\mathbf{M}} K^F(x,y)f(y)dy,$$

where

$$K^F(x,y) = \sum_{j=0}^{\infty} F(\lambda_j)\varphi_j(x)\varphi_j(y).$$

We will be especially interested in operators of the form $F(t^2\mathcal{L})$, where F is a Schwartz function and $t > 0$. The corresponding kernel will be denoted as $K_t^F(x, y)$ and

$$K_t^F(x, y) = \sum_{j=0}^{\infty} F(t^2\lambda_j)\varphi_j(x)\varphi_j(y).$$

Variable t here is a kind of scaling parameter.

Choose a function $F \in C_c^\infty$, supported in the interval $[2^{-2}, 2^4]$ such that

$$\sum_{j=-\infty}^{\infty} |F(2^{-2j}s)|^2 = 1 \tag{6}$$

for all $s > 0$.
(For example, we could choose a smooth function Φ on $\mathbb{R}^+$ with $0 \leq \Phi \leq 1$, with $\Phi \equiv 1$ in $[0, 2^{-2}]$ and with $\Phi = 0$ in $[2^2, \infty)$, and let $F(t) = [\Phi(t/2^2) - \Phi(t)]^{1/2}$ for $t > 0$.)

Note that the eigenspace for $\mathcal{L}$ corresponding to the eigenvalue $\lambda_0 = 0$ is the space of constant functions. Let P be the projection in $L_2(\mathbf{M})$ onto the space of constant functions. Using digonalization of $\mathcal{L}$ one can obtain

$$\sum_{j=-\infty}^{\infty} |F|^2(2^{-2j}\mathcal{L}) = I - P, \tag{7}$$

where the sum converges strongly on $L_2(\mathbf{M})$.

Pick a function $f \in L_2(\mathbf{M})$. We apply (7) to f and take the inner product with f. We find

$$\sum_{j=-\infty}^{\infty} \|F(2^{-2j}\mathcal{L})f\|_2^2 = \|(I-P)f\|_2^2 \tag{8}$$

Expand $f \in L_2(\mathbf{M})$ in terms of eigenfunctions of $\mathcal{L}$

$$f = \sum_m c_m \varphi_m, \quad c_m = < f, \varphi_m > .$$

Then one has

$$F(2^{-2j}\mathcal{L})f = \sum_m F(2^{-2j}\lambda_m) c_m \varphi_m.$$

Since for every j function $F(2^{-2j}s)$ is supported in the interval $[2^{2j+2}, 2^{2j+4}]$ the function $F(2^{-2j}\mathcal{L})f$ is bandlimited and belongs to $\mathbf{E}_{2^{2j+4}}(\mathcal{L})$.

We also have $\overline{F(2^{-2j}\mathcal{L})f} \in \mathbf{E}_{2^{2j+4}}(\mathcal{L})$
According to the product rule, the product $|F(2^{-2j}\mathcal{L})f|^2$ of these two functions is also bandlimited and

$$|F(2^{-2j}\mathcal{L})f|^2 \in \mathbf{E}_{4d2^{2j+4}}(\mathcal{L}),$$

where $d = dim\, G$, $\mathbf{M} = G/K$.

To summarize, we proved that for every $f \in L_2(\mathbf{M})$ we have decomposition

$$\sum_{j=-\infty}^{\infty} \|F(2^{-2j}\mathcal{L})f\|_2^2 = \|(I-P)f\|_2^2 \tag{9}$$

where for every j

$$\|F(2^{-2j}\mathcal{L})f\|_2^2 = \int_{\mathbf{M}} |F(2^{-2j}\mathcal{L})f|^2(x)dx \tag{10}$$

and where

$$|F(2^{-2j}\mathcal{L})f|^2 \in \mathbf{E}_{4d2^{2j+4}}(\mathcal{L}). \tag{11}$$

The next objective is to perform a discretization step. According to our result about cubature formula there exists a constant $a_0 > 0$ such that for all integer j if

$$\rho_j = a_0(4d2^{2j+4} + 1)^{-1/2} \tag{12}$$

then for any ρ_j-lattice M_{ρ_j} one can find coefficients $b_k^{(j)}$ with

$$b_k^{(j)} \sim \rho_j^n, \tag{13}$$

for which the following exact cubature formula holds

$$\|F(2^{-2j}\mathcal{L})f\|_2^2 = \sum_{k=1}^{\mathcal{N}_j} b_k^{(j)} |[F(2^{-2j}\mathcal{L})f](x_k^{(j)})|^2, \tag{14}$$

where $x_k^{(j)} \in M_{\rho_j}$, $(k = 1, \ldots, \mathcal{N}_j = N(M_{\rho_j}))$

Now, for $t > 0$, let K_t^F be the kernel of $F(t^2\mathcal{L})$, so that, for $f \in L_2(\mathbf{M})$,

$$[F(t^2\mathcal{L})Ff(x) = \int_{\mathbf{M}} K_t(x,y)F(y)dy. \tag{15}$$

For $x, y \in \mathbf{M}$, we have

$$K_t^F(x,y) = \sum_m F(t^2\lambda_m)\varphi_m(x)\overline{\varphi}_m(y). \tag{16}$$

Corresponding to each $x_k^{(j)}$ we now define the functions

$$\psi_k^{(j)}(y) = \overline{K_{a^{-j}}^{F}}(x_k^{(j)}, y) = \sum_m \overline{F}(a^{-2j}\lambda_m)\overline{\varphi}_m(x_k^{(j)})\varphi_m(y), \tag{17}$$

$$\Psi_k^{(j)} = \sqrt{b_k^{(j)}}\psi_k^{(j)}. \tag{18}$$

We find that for all $f \in L_2(\mathbf{M})$,

$$\|(I-P)f\|_2^2 = \sum_{j,k} |\langle f, \Psi_k^{(j)} \rangle|^2. \tag{19}$$

Note that, by (17) and (18), and the fact that $F(0) = 0$, each $\Psi_k^{(j)} \in (I-P)L_2(\mathbf{M})$.

Thus the $\Psi_k^{(j)}$ form a Parseval frame (i.e. normalized tight frame) for $(I - P)L_2(\mathbf{M})$.
Note also that each $\Psi_k^{(j)}$ is a finite linear combination of eigenfunctions of $\mathcal{L}$.

Moreover, since F vanishes on $[2^4, \infty)$, we have $\Psi_k^{(j)} \equiv 0$ once $2^{-2j}\lambda_1 \geq 2^4$. Thus, for some Ω
specifically $\Omega = [(\log_2 \lambda_1/2) - 1]$,
we have

$$\Psi_k^{(j)} \equiv 0, \quad j < \Omega. \tag{20}$$

Note that, by out choice of ρ (see (12)), for $j \geq \Omega$, we have

$$\rho_j \sim 2^{-j}, \tag{21}$$

in the sense that the ratio of these quantities is bounded above and below by positive constants.

By gereral frame theory, if $f \in L_2(\mathbf{M})$, we have

$$(I - P)f = \sum_{j=\Omega}^{\infty} \sum_{k} \langle f, \Psi_k^{(j)} \rangle \Psi_k^{(j)} = \sum_{j=\Omega}^{\infty} \sum_{k} b_k^{(j)} \langle f, \psi_k^{(j)} \rangle \psi_k^{(j)}, \quad (22)$$

with convergence in $L_2(\mathbf{M})$.

Concerning localization of $\Psi_k^{(j)}$ one can prove the following

Theorem

(Near-diagonal localization)
Assume that $F \in \mathcal{S}(\mathbf{R})$ *is a Schwartz functions on* $\mathbf{R}$.
For $t > 0$, *let* $K_t^F(x, y)$ *be the kernel of* $F(t^2\mathcal{L})$.
If $F(0) = 0$ *then for every integer* $N \geq 0$, *there exists* C_N *such that*

$$|K_t(x, y)| \leq C_N t^{N-n} \left(1 + dist\,(x, y)\right)^{-N}, \quad n = dim\,\mathbf{M},$$

for all $t > 0$ *and all* $x, y \in \mathbf{M}$.

THANK YOU FOR COMING!