Uncertainty principles for the Schrödinger equation on Riemannian symmetric spaces of the noncompact type

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Joint work with Maddala Sundari

Geometric Analysis on Euclidean and Homogeneous Spaces
Tufts University

January 8–9, 2012

Qualitative uncertainty principle

= a theorem which allows to conclude that $f = 0$ by giving quantitative conditions on f and its Fourier transform $\widehat{f}$

In this talk:
Initial value problem for the time-dependent Schrödinger equation on a Riemannian symmetric space of the noncompact type X:

$$\begin{aligned} i\partial_t u(t,x) + \Delta u(t,x) &= 0 \\ u(0,x) &= f(x) \end{aligned} \tag{S}$$

where Δ denotes the Laplace-Beltrami operator on X and $f \in L^2(X)$.

We prove:
The solution to (S) is identically zero at all times t whenever the initial condition f and the solution at a time $t_0 > 0$ are simultaneously "very rapidly decreasing".

$\rightsquigarrow$ This is an uncertainty principle because of the relation linking the Fourier-Helgason transforms of f and of the solution $u_t = u(t, \cdot)$

The condition of rapid decrease we consider is of Beurling type.

Previous work:

Schrödinger equation on $\mathbb{R}^n$:
Escauriaza, Kenig, Ponce, Vega (& Cowling) (2006, 2008, 2010, 2011)
Also nonlinear Schrödinger equation, with a time-dependent potential,...
Uncertainty conditions on the initial condition f and the solution $u(t_0, \cdot)$ at a time t_0 of different types: Hardy, Morgan, Beurling...

On nilpotent Lie groups:
Ben Said & Thangavelu (2010) for the Heisenberg group
Uncertainty conditions of Hardy type

On Riemannian symmetric spaces G/K of the noncompact type:
Chanillo (2007) for G complex and K-invariant functions
Uncertainty conditions of Hardy type

Beurling-type uncertainty principles on $\mathbb{R}^n$

Theorem (Beurling's theorem on $\mathbb{R}$, Hörmander (1991))

Let $f \in L^1(\mathbb{R})$. If

$$\int_{\mathbb{R}} \int_{\mathbb{R}} |f(x)||\widehat{f}(y)|e^{|xy|}\, dx\, dy < \infty$$

then $f = 0$ almost everywhere.

A higher dimensional version:

Theorem (Beurling's theorem on $\mathbb{R}^n$, Bagchi & Ray (1998))

Let $f \in L^1(\mathbb{R}^n)$. Suppose that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)||\widehat{f}(y)|e^{|x||y|}\, dx\, dy < \infty\,.$$

Then $f = 0$ almost everywhere.

Sharpest version: Bonami, Demange & Jaming (2003)

Beurling's theorem

$\Downarrow$

Gelfand-Shilov type

Let $f \in L^2(\mathbb{R}^n)$ *and* $1 < p, q < \infty$ *with* $1/p + 1/q = 1$. *Suppose* $\exists\ a > 0$, $b > 0$ *so that*

$$\int_{\mathbb{R}^n} |f(x)| e^{\frac{a^p}{p}|x|^p}\, dx < \infty \quad \textit{and} \quad \int_{\mathbb{R}^n} |\widehat{f}(x)| e^{\frac{b^q}{q}|x|^q}\, dx < \infty$$

If $ab \geq 1$, *then* $f = 0$ *almost everywhere.*

$\Downarrow$

Cowling-Price type

Let $f \in L^2(\mathbb{R}^n)$ *and let* $1 \leq p, q < \infty$. *Suppose* $\exists\ a > 0$, $b > 0$ *so that*

$$\int_{\mathbb{R}^n} (|f(x)| e^{a|x|^2})^p dx < \infty \quad \textit{and} \quad \int_{\mathbb{R}^n} (|\widehat{f}(x)| e^{b|x|^2})^q dx < \infty$$

If $ab > 1/4$, *then* $f = 0$ *almost everywhere.*

$\Downarrow$

Hardy type

Let f *be measurable on* $\mathbb{R}^n$. *Suppose* $\exists\ a > 0$, $b > 0$ *so that*

$$|f(x)| \leq e^{-a|x|^2} \quad \textit{and} \quad |\widehat{f}(x)| \leq e^{-b|x|^2}$$

If $ab > 1/4$, *then* $f = 0$ *almost everywhere.*

The damped Schrödinger equation on $\mathbb{R}^n$

Initial value problem for the time-dependent damped Schrödinger equation on $\mathbb{R}^n$:

$$\begin{aligned} i\partial_t u(t,x) + (\Delta - c)u(t,x) &= 0 \\ u(0,x) &= f(x) \end{aligned} \qquad (S_c)$$

where:

Δ = Laplace operator on $\mathbb{R}^n$

$c \in \mathbb{R}$ = the damping parameter

Suppose $f \in L^2(\mathbb{R}^n)$.

Take Fourier transform in the x-variable of both sides of the equations.

This gives rise to an ODE in the t-variable that can be solved directly.

$\rightsquigarrow$ If $f \in L^2(\mathbb{R}^n)$, then $\exists$ unique $u \in C(\mathbb{R} : L^2(\mathbb{R}^n))$ satisfying (S_c) in the sense of distributions. The solution $u_t(x) = u(t,x)$ is characterized by the equation

$$\widehat{u_t}(\lambda) = e^{-i(|\lambda|^2+c)t}\,\widehat{f}(\lambda)\,. \qquad (*)$$

More generally: (*) admits unique solution $u_t \in S'(\mathbb{R}^n)$ if $f \in \mathcal{S}'(\mathbb{R}^n)$.

Explicitly: for $t \neq 0$ and $f \in L^2(\mathbb{R}^n)$ (or $f \in L^1(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$):

$$u(t,x) = (2\pi)^{n/2}(2|t|)^{-n/2} e^{-ict} e^{-\pi i(\operatorname{sign} t)n/4} e^{i\frac{|x|^2}{4t}}\ \widehat{h_t}\left(\frac{x}{2t}\right)$$

where

$$h_t(y) = e^{i\frac{|y|^2}{4t}}\ f(y)\,.$$

Remark: If exists $t_0 > 0$ so that $h_{t_0} = 0$ (i.e. $u_{t_0} = 0$), then $f = 0$

Theorem (Cowling et al., P.-Sundari)

Let $u_t(x) = u(t,x)$ be the solution to the damped Schrödinger equation (S_c) with initial condition $f \in L^2(\mathbb{R}^n)$ or $f \in L^1(\mathbb{R}^n)$. If there is $t_0 > 0$ so that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||u(t_0,y)| e^{\frac{|x||y|}{2t_0}}\, dx\, dy < \infty \qquad \text{(BS)}$$

then $f = 0$. Hence $u(t,\cdot) = 0$ for all $t \in \mathbb{R}$.

Proof. Suppose first $f \in L^1(\mathbb{R}^n)$. Then:

$$+\infty > \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |f(x)||u(t_0,y)| e^{\frac{|x||y|}{2t_0}}\, dx\, dy = \frac{1}{(2t_0)^{n/2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|h_{t_0}(x)\widehat{h_{t_0}}\left(\frac{y}{2t_0}\right)\right| e^{\frac{|x||y|}{2t_0}}\, dx\, dy$$
$$= (2t_0)^{n/2} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |h_{t_0}(x)||\widehat{h_{t_0}}(y)| e^{|x||y|}\, dx\, dy$$

Hence $f = e^{-i\frac{|\cdot|^2}{4t_0}} h_{t_0} = 0$ by Beurling's theorem.
If $f \in L^2(\mathbb{R}^n)$ satisfies (BS), then $f \in L^1(\mathbb{R}^n)$. Indeed:

$$|u(t_0,y)| \int_{\mathbb{R}^n} |f(x)| e^{\frac{|x||y|}{2t_0}}\, dx < +\infty \qquad \text{for almost all } y\,.$$

If $u(t_0,y) = 0$ a.e. y, then $f = 0$.
If not, $\exists y_0$ such that $\|f\|_1 \le \int_{\mathbb{R}^n} |f(x)| e^{\frac{|x||y_0|}{2t_0}}\, dx < +\infty$. □

The Schrödinger equation on $X = G/K$

$X = G/K$ Riemannian symmetric space of the noncompact type
where G = noncompact connected semisimple Lie group with finite center
K = maximal compact subgroup of G
Δ = Laplace-Beltrami operator on X

Initial value problem for the time-dependent Schrödinger equation on X with $f \in L^2(X)$:

$$\begin{aligned} i\partial_t u(t,x) + \Delta u(t,x) &= 0 \\ u(0,x) &= f(x) \end{aligned} \qquad \text{(S)}$$

$o = eK$ the base point of X
$\sigma(x) = d(x,o)$ = distance from $x \in X$ to o wrt the Riemannian metric d
$\Xi(x)$ = the (elementary) spherical function of spectral parameter 0

Theorem

Let $u(t,x) \in C(\mathbb{R} : L^2(X))$ denote the solution of (S) with initial condition $f \in L^2(X)$. If there is a time $t_0 > 0$ so that

$$\int_X \int_X |f(x)||u(t_0,y)|\Xi(x)\Xi(y)e^{\frac{\sigma(x)\sigma(y)}{2t_0}}\,dx\,dy < \infty\,, \qquad \text{(BS)}$$

then $u(t,\cdot) = 0$ for all $t \in \mathbb{R}$.

Idea of proof: Helgason-Fourier transform + (careful) Euclidean reduction via Radon transform.

The Helgason-Fourier and the Radon transforms

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G

where $\mathfrak{k}$ = Lie algebra of K

$\mathfrak{a} \subset \mathfrak{p}$ max abelian subspace (Cartan subspace)

$A = \exp \mathfrak{a}$ abelian subgroup of G

$G = KAN$ Iwasawa decomposition

$H(g)$ = Iwasawa projection of $g \in G$ on $\mathfrak{a}$, i.e. $g = k(g) \exp H(g) n(g)$

M = centralizer of A in K, and $B = K/M$

$A : X \times B \to \mathfrak{a}$ by $A(gK, kM) := -H(g^{-1}k)$

$\Sigma \subset \mathfrak{a}^*$ roots of $(\mathfrak{g}, \mathfrak{a})$

Σ^+ = choice of positive roots

m_α = multiplicity of the root $\alpha \in \Sigma$

$\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$

For $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$ and $b \in B$, define: $e_{\lambda,b}(x) = e^{(i\lambda+\rho)(A(x,b))}, \qquad x \in X.$

The *Helgason-Fourier transform* of a (sufficiently regular) function $f : X \to \mathbb{C}$ is the function $\mathcal{F}f : \mathfrak{a}^*_{\mathbb{C}} \times B \to \mathbb{C}$ defined by

$$\mathcal{F}f(\lambda, b) = \int_X f(x) e_{-\lambda,b}(x)\, dx = \int_{AN} f(kan \cdot o) e^{(-i\lambda+\rho)(\log a)}\, da\, dn$$

($b = kM$; Haar measures da and dn suitably normalized)

Theorem (Helgason, 1965; Eguchi, 1980)

1. (Plancherel theorem) $\mathcal{F}$ *extends to an isometry of* $L^2(X)$ *onto* $L^2(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2} d\lambda\, db)$. *Here c is Harish-Chandra's c-function.*
2. (Inversion formula) *If* $f \in L^1(X)$ *and* $\mathcal{F}f \in L^1(\mathfrak{a}_+^* \times B, |c(\lambda)|^{-2} d\lambda\, db)$, *then*

$$f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^* \times B} \mathcal{F}f(\lambda, b) e_{\lambda,b}(x) \frac{d\lambda\, db}{|c(\lambda)|^2} \qquad a.e.\ x \in X$$

The *(modified) Radon transform* of a (sufficiently regular) function $f : X \to \mathbb{C}$ is the function $Rf : B \times A \to \mathbb{C}$ defined by

$$Rf(b, a) = e^{\rho(\log a)} \int_N f(kan \cdot o)\, dn\,, \qquad b = kM\,.$$

The *Euclidean Fourier transform* of a (sufficiently regular) function $f : A \to \mathbb{C}$ is the function $\mathcal{F}_A f : \mathfrak{a}^* \to \mathbb{C}$ defined by

$$(\mathcal{F}_A f)(\lambda) := \int_A f(a) e^{-i\lambda(\log a)}\, da\,, \qquad \lambda \in \mathfrak{a}^*\,.$$

$\mathcal{F}$ and R (on suitable classes of functions on X) are linked by the Euclidean transform.
For instance: if $f \in L^1(X)$, then $Rf \in L^1(B \times A)$ and

$$\mathcal{F}f(\lambda, b) = \mathcal{F}_A\big(Rf(b, \cdot)\big)(\lambda) = \int_A Rf(b, a) e^{-i\lambda(\log a)}\, da\,.$$

for almost all $b \in B$ and all $\lambda \in \mathfrak{a}^*$.

Solution of the Schrödinger equation on $X = G/K$

$$\begin{aligned} i\partial_t u(t,x) + \Delta u(t,x) &= 0 \\ u(0,x) &= f(x) \in L^2(X) \end{aligned} \qquad (S)$$

The Fourier-transform method used on $\mathbb{R}^n$ carries out to (S) when the Fourier transform is replaced by the Helgason-Fourier transform $\mathcal{F}$.

The functions $e_{\lambda,b}$ appearing in the definition of $\mathcal{F}$ are eigenfunctions of Δ

Plancherel: $\mathcal{F}$ is a unitary equivalence between Δ on $L^2(G/K)$ and the multiplication operator M on $L^2(\mathfrak{a}^* \times B, \frac{d\lambda\, db}{|c(\lambda)|^2})$ defined by $(Mf)(\lambda,b) = -(|\lambda|^2 + |\rho|^2)f(\lambda,b)$.

Hence: there is a unique $u_t(x) = u(t,x) \in C(\mathbb{R} : L^2(X))$ satisfying (S) in the sense of distributions. It is characterized by the equation

$$(\mathcal{F}u_t)(\lambda,b) = e^{-i(|\lambda|^2+|\rho|^2)t}\mathcal{F}f(\lambda,b)\,.$$

i.e.

$$u_t = \mathcal{F}^{-1}\left(e^{-i(|\lambda|^2+|\rho|^2)t}\mathcal{F}f\right)\,.$$

Remark: If $f=0$, then $u_t = 0$. Conversely, if $\exists\ t_0 \in \mathbb{R}$ so that $u_{t_0} = 0$, then $e^{-i(|\lambda|^2+|\rho|^2)t_0}\mathcal{F}f(\lambda,b) = 0$ for all (λ,b). Hence $f=0$ as $\mathcal{F}$ is injective on $L^2(X)$.

Fourier analysis on $B = K/M$

$\widehat{K}_M =$ (equiv classes of) irreducible unitary reps of K with nonzero M-fixed vectors

$V_\delta =$ space of $\delta \in \widehat{K}_M$
inner product $\langle \cdot, \cdot \rangle$

$d(\delta) = \dim V_\delta$

$\{v_1^\delta, \dots, v_{d(\delta)}^\delta\} =$ ON basis of V_δ

The *Fourier coefficients* of $F \in L^1(B)$ are

$$F_{i,j}^\delta = \int_K \langle \delta(k^{-1}) v_i^\delta, v_j^\delta \rangle F(kM)\, dk\,.$$

Then:

$$F = 0 \text{ if and only if } F_{i,j}^\delta = 0 \text{ for all } \delta \in \widehat{K}_M \text{ and all } i, j = 1, \dots, d(\delta).$$

The space $L^1(X)_C$ (with $C \geq 0$)

For measurable $h : X \to \mathbb{C}$ set: $\|h\|_{1,C} := \int_X |h(x)|\Xi(x)e^{C\sigma(x)}\, dx$

$L^1(X)_C$ = (equiv classes of a.e. equal) functions $h : X \to \mathbb{C}$ with $\|h\|_{1,C} < \infty$.

Motivation: If f, u_{t_0} satisfy $\int_X \int_X |f(x)||u(t_0, y)|\Xi(x)\Xi(y)e^{\sigma(x)\sigma(y)/(2t_0)}\, dx\, dy < \infty$,
then $\exists C > 0$ so that $f \in L^1(X)_C$. Likewise for u_{t_0}.

Remark: $L^1(X)_C \subset L^1(X)$ if $C >> 0$ (e.g. $C \geq |\rho|$).

Lemma

Let $h \in L^1(X)_C$. Then:

1. *The Radon transform Rh is a.e. defined and in $L^1(B \times A, db\, da)$*
2. *For all $\lambda \in \mathfrak{a}^*$, the Helgason-Fourier transform $\mathcal{F}h(\lambda, \cdot)$ is a.e. defined and in $L^1(B)$*

Proof for $C = 0$.

$$\infty > \|h\|_{1,C=0} = \int_G |h(g \cdot o)|e^{-\rho(H(g))}\, dg \quad \text{(by } \Xi(g) = \textstyle\int_K e^{-\rho(H(gk))} dk \text{ \& } K\text{-inv of } \sigma \text{ and } h)$$

$$\geq \int_K \int_A \left(e^{\rho(\log a)} \int_N |h(kan \cdot o)|\, dn \right) da\, dk \quad \text{(by Iwasawa decomp \& } \sigma(an) \geq \sigma(a))$$

$$\geq \int_B \int_A |(Rh)(b, a)|\, da\, db$$

Moreover: $|\mathcal{F}h(\lambda, b)| \leq \int_A \int_N |h(kan \cdot o)|e^{\rho(\log a)}\, da\, dn \leq \int_A |(Rh)(b, a)|\, da \quad (b = kM)$ □

Consequence: If $h \in L^1(X)_C$ then $Rh(b,\cdot) \in L^1(A)$ for almost all $b \in B$

$\Rightarrow$ we can consider $\mathcal{F}_A\big(Rh(b,\cdot)\big)$

Lemma

$\mathcal{F} = \mathcal{F}_A \circ R$ *on* $L^1(X)_C$.

Proof (Sketch).

For $\delta \in \widehat{K}_M$ and $i,j = 1,\ldots,d(\delta)$ define:

$$(Rh)^{\delta}_{i,j}(a) := \big(Rh(\cdot,a)\big)^{\delta}_{i,j} \qquad \text{for almost all } a \in A$$

$$(\mathcal{F}h)^{\delta}_{i,j}(\lambda) := \big(\mathcal{F}h(\lambda,\cdot)\big)^{\delta}_{i,j} \qquad \text{for all } \lambda \in \mathfrak{a}^*$$

Get: $\mathcal{F}_A\big((Rh)^{\delta}_{i,j}(a)\big) : \mathfrak{a}^* \to \mathbb{C}$ and $(\mathcal{F}h)^{\delta}_{i,j} : \mathfrak{a}^* \to \mathbb{C}$.

The functions $L^1(X)_C \to L^\infty(\mathfrak{a}^*)$ defined by $\begin{cases} h \mapsto \mathcal{F}_A\big((Rh)^{\delta}_{i,j}(a)\big) \\ h \mapsto (\mathcal{F}h)^{\delta}_{i,j} \end{cases}$

are continuous and equal on $C_c^\infty(X)$.

For all $\lambda \in \mathfrak{a}^*$:

$$\mathcal{F}h(\lambda,\cdot)^{\delta}_{i,j} = (\mathcal{F}h)^{\delta}_{i,j}(\lambda) = \mathcal{F}_A\big((Rh)^{\delta}_{i,j}(a)\big)(\lambda) = \big(\mathcal{F}_A(Rh)(\lambda,\cdot)\big)^{\delta}_{i,j}$$

$$\Rightarrow \quad \mathcal{F}h(\lambda,\cdot) = \mathcal{F}_A(Rh)(\lambda,\cdot)$$

□

Proof of the main theorem:

Theorem

Let $u(t,x) \in C(\mathbb{R} : L^2(X))$ denote the solution of (S) with initial condition $f \in L^2(X)$. If there is a time $t_0 > 0$ so that

$$\int_X \int_X |f(x)||u(t_0,y)|\Xi(x)\Xi(y)e^{\frac{\sigma(x)\sigma(y)}{2t_0}}\,dx\,dy < \infty\,, \qquad \text{(BS)}$$

then $u(t,\cdot) = 0$ for all $t \in \mathbb{R}$.

If (BS) holds, then $f \in L^1(X)_C$ for some $C > 0$.
For $\delta \in \widehat{K}_M$ and $i,j = 1,\dots,d(\delta)$, condition (BS) yields

$$\int_A \int_A |(Rf)^\delta_{i,j}(a_1)||(Ru_{t_0})^\delta_{i,j}(a_2)|e^{\frac{|\log a_1||\log a_2|}{2t_0}}\,da_1\,da_2 < \infty$$

Using $\mathcal{F} = \mathcal{F}_A \circ R$, we obtain from $(\mathcal{F}u_t)(\lambda,b) = e^{-i(|\lambda|^2+|\rho|^2)t}\mathcal{F}f(\lambda,b)$:

$$\mathcal{F}_A\big((Ru_t)^\delta_{i,j}\big)(\lambda) = e^{-i(|\lambda|^2+|\rho|^2)t}\mathcal{F}_A\big((Rf)^\delta_{i,j}\big)(\lambda)\,.$$

Euclidean case: $(Rf)^\delta_{i,j} = 0$ a.e. on A. Conclusion: $Rf = 0$.
Hence $\mathcal{F}f = \mathcal{F}_A(Rf) = 0$. Thus $f = 0$ as $\mathcal{F}$ injective on $L^2(X)$. □

Corollary

Let $u(t,x) \in C(\mathbb{R} : L^2(X))$ be the solution of (S) with initial condition $f \in L^2(X)$.
Suppose that f has compact support.
If $\exists\ t_0 > 0$ so that $u(t_0,\cdot)$ has compact support, then $u(t,\cdot) = 0$ for all $t \in \mathbb{R}$.

Applications: other uncertaninty principles

Let $u(t,x) \in C(\mathbb{R} : L^2(X))$ be the solution of (S) with initial condition $f \in L^2(X)$.

Corollary (Gelfand-Shilov type)

Let $1 < p < \infty$ *and* $1/p + 1/q = 1$. *Suppose* $\exists\ \alpha > 0$, $\beta > 0$ *and a time* $t_0 > 0$ *so that*

$$\int_X |f(x)|\Xi(x) e^{\frac{\alpha^p}{p}\sigma^p(x)}\,dx < \infty \qquad \text{and} \qquad \int_X |u(t_0,x)|\Xi(x) e^{\frac{\beta^q}{q}\sigma^q(x)}\,dx < \infty\,.$$

If $2t_0\alpha\beta \geq 1$, *then* $f = 0$ *and hence* $u(t,\cdot) = 0$ *for all* $t \in \mathbb{R}$.

Corollary (Cowling-Price type)

Let $1 \leq p, q < \infty$. *Suppose* $\exists\ \alpha > 0$, $\beta > 0$ *and a time* $t_0 > 0$ *so that*

$$\int_X \left(|f(x)|e^{a\sigma^2(x)}\right)^p\,dx < \infty \qquad \text{and} \qquad \int_X \left(|u(t_0,x)|e^{b\sigma^2(x)}\right)^q\,dx < \infty\,.$$

If $16t_0^2 ab > 1$, *then* $f = 0$ *and hence* $u(t,\cdot) = 0$ *for all* $t \in \mathbb{R}$.

Corollary (Hardy type; S.Chanillo, 2007, for *G* complex & *f* *K*-inv)

Suppose $\exists\ A > 0$, $\alpha > 0$ *so that* $|f(x)| \leq A\,e^{-\alpha\sigma^2(x)}$ *for all* $x \in X$.
Suppose $\exists\ t_0 > 0$ *and* $B > 0$, $\beta > 0$ *so that* $|u(t_0,x)| \leq B\,e^{-\beta\sigma^2(x)}$ *for all* $x \in X$.
If $16\alpha\beta t_0^2 > 1$, *then* $u(t,\cdot) = 0$ *for all* $t \in \mathbb{R}$.