RANGE DESCRIPTION FOR SPHERICAL MEAN TRANSFORM ON SPACES OF CONSTANT CURVATURE

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GEOMETRIC ANALYSIS ON EUCLIDEAN AND HOMOGENEOUS SPACES

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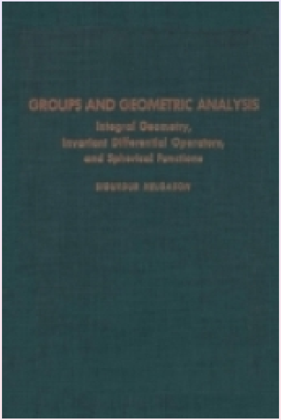
SPHERICAL MEAN TRANSFORM



Thank you for the wonderful book, Professor Helgason!

Outline

1. Spherical mean transform on $\mathbb{R}^n$
2. Spherical mean transform on $\mathbb{H}^n$
3. Spherical mean transform on $\mathbb{S}^n$

SPHERICAL MEAN TRANSFORM

- Let $f \in C(\mathbb{R}^n)$, $\mathcal{R}(f)(x,t)$ is the mean value of f on the sphere $S(x,t)$:

$$\mathcal{R}(f)(x,t) = \frac{1}{|S(x,t)|} \int\limits_{S(x,t)} f(y) d\sigma(y).$$

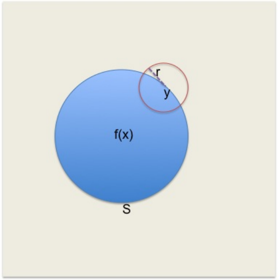
- Arises in PDEs, approximation theory, biomedical/geophysical imaging.
- Classics: Courant-Hilbert (Methods of Mathematical Physics, Vol. 2), F. John (Plane Waves and Spherical Means Applied to Partial Differential Equations), S. Helgason.
- The function $G(x,t) = \mathcal{R}(f)(x,t)$ satisfies the Darboux equation:

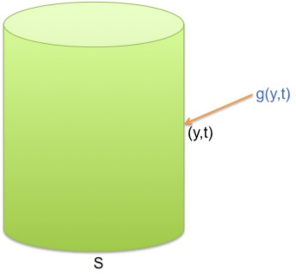
$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t} \frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, \\ G(x,0) = f(x), \ G_t(x,0) = 0. \end{cases}$$

- Conversely, if G satisfies the above equation, $G = \mathcal{R}(f)$.

RESTRICTED SPHERICAL MEAN TRANSFORM

- Let $S \subset \mathbb{R}^n$ a hyper-surface.
- $\mathcal{R}_S$ restriction of $\mathcal{R}$ to all the spheres whose centers lie on S.
- Goal: characterize the range of $\mathcal{R}_S(f)$.
- Too general to deal with.
- Assumptions: S is the unit sphere, $f \in C_0^\infty(\mathbb{R}^n)$ and *supp*$(f) \subset \overline{B}$.





Problem

Characterize all the functions g defined on $S \times \mathbb{R}_+$ such that $g = \mathcal{R}_S(f)$ for some function $f \in C_0^\infty(\overline{B})$.

NECESSARY CONDITIONS: SMOOTHNESS & SUPPORT, ORTHOGONALITY

a) Smoothness & Support: $g \in C_0^\infty([0,2])$
- $g \in C_0^\infty(S \times \mathbb{R}_{\geq 0})$,
- $g(x,t)$ vanishes up to infinite order at $t = 0$,
- $g(x,t) = 0$ for $t \geq 2$.

b) Orthogonality condition:
- Recall

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, \ x \in \mathbb{R}^n, \\ G(x,0) = f(x), \ G_t(x,t) = 0, \ x \in \mathbb{R}^n. \end{cases}$$

- Let $-\Delta\varphi(x) = \lambda_k^2 \varphi(x), \ \varphi|_S = 0$, and $j_{\frac{n}{2}-1}$-Bessel function of order $\frac{n}{2} - 1$.
- Then $u(x,t) = \varphi(x) j_{\frac{n}{2}-1}(\lambda_k t)$ solves Darboux equation in $B \times \mathbb{R}_+$ with $u|_{S \times \mathbb{R}_+} = 0$.
- Multiply the equation by $u(x,t)t^{n-1}$ and taking integration by parts:

$$\int\limits_S \int\limits_0^\infty g(y,t) \frac{\partial \varphi(x)}{\partial \nu} j_{\frac{n}{2}-1}(\lambda_k t) dt d\sigma(y) = 0.$$

NECESSARY CONDITIONS: MOMENT CONDITION

c) Moment condition: M_k is a polynomial of degree at most $2k$.

- Recall

$$\mathcal{R}(f)(x,t) = \frac{1}{|S(x,t)|} \int\limits_{S(x,t)} f(y)\, d\sigma(y).$$

- Hence,

$$\begin{aligned} M_0(x) &:= \int\limits_0^\infty t^{n-1}\mathcal{R}(f)(x,t) = \int\limits_0^\infty \frac{t^{n-1}}{|S(x,t)|} \int\limits_{S(x,t)} f(y)\, d\sigma(y) \\ &= \frac{1}{\omega_n} \int\limits_0^\infty \int\limits_{S(x,t)} f(y)\, d\sigma(y)\, dt = \int\limits_{\mathbb{R}^n} f(y)\, dy. \end{aligned}$$

- Also:

$$M_k(x) := \frac{1}{\omega_n} \int\limits_0^\infty \int\limits_{S(x,t)} t^{2k+n-1} f(y)\, d\sigma(y)\, dt = \int\limits_{\mathbb{R}^n} |x-y|^{2k} f(y)\, dy = P_k(x).$$

Here, P_k is a polynomial of degree at most $2k$.

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SUFFICIENCY

1. Ambartsoumian and Kuchment (SIAM J. on Math. An., 2007): a), b), c) sufficient when $n = 2$.
2. Finch and Rakesh (Inverse Problem, 2006): a) and b) sufficient when n odd.
3. Agranovsky, Kuchment, and Quinto (J. of Functional Analysis, 2007):
   - a)+ b)+ c) sufficient for all n.
   - a) + b) sufficient when n odd.
4. Agranovsky, Finch, and Kuchment (Inverse Problems and Imaging, 2009): $a) + b) \Rightarrow c)$.
5. Agranovsky and Nguyen (J. d'Analyse Mathematique, 2011): a) + b) sufficient (combinatorial arguments).
6. Other works: Patch-'04, Palamodov-'09.

Theorem (Agranovsky-Finch-Kuchment-'09, Agranovsky-Nguyen-'11)

Conditions (a) Smoothness & Support and (b) Orthogonality are necessary and sufficient for the range description for $\mathcal{R}_S$.

PROOF

Restatement: Under the conditions (a) and (b), there is a global solution $G(x,t)$:

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, x \in \mathbb{R}^n, \\ G(x,0) = f(x),\ G_t(x,t) = 0,\ x \in \mathbb{R}^n, \\ G(y,t) = g(y,t),\ (y,t) \in S \times \mathbb{R}_+, \end{cases} \qquad (1)$$

for some $f \in C_0^\infty(\overline{B})$.

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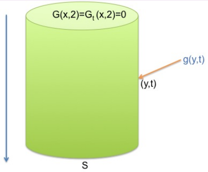
$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0,\ x \in B, \\ G(x,2) = 0,\ G_t(x,2) = 0,\ x \in \mathbb{R}^n, \\ G(y,t) = g(y,t),\ (y,t) \in S \times \mathbb{R}_+, \end{cases}$$



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2. Step 2: The function f^* vanishes up to infinite order at the boundary S of B.
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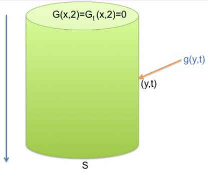

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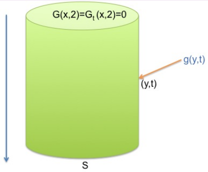

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PROOF of step 2

Theorem (Agranovsky-N., d'Analyse, 2011)

Assume that

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, \ (x,t) \in B \times \mathbb{R}_+, \\ G(x,0) = f^*(x), \ G_t(x,0) = 0, \ x \in B, \\ G(y,t) = g(y,t), \ (y,t) \in S \times \mathbb{R}_+, \end{cases}$$

has a unique smooth solution $G(x,t)$ such that $G(x,2) = G_t(x,2) = 0$. Then, f^* vanishes up to infinite order on S.

Let $g \in C_0^\infty(S \times [0,2])$. Assume $f^*(x) = f_m(r) r^m Y_m(\theta)$. Show f_m vanishes up to infinite order at $r = 1$.

- Iterating the equation k-times:

$$\Delta^k f(x) = \left(\frac{\partial^2}{\partial t^2} + \frac{n-1}{t}\frac{\partial}{\partial t}\right)^k g(x,0) = 0, \ \forall x \in S.$$

  This implies

$$\left(\frac{d^2}{dr^2} + \frac{n-1+2m}{r}\frac{d}{dr}\right)^k f_m(r) = 0, \ \forall k \geq 0.$$

PROOF of step 2

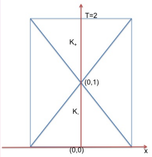
Theorem

Assume that

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, \ x \in B, \\ G(x,0) = f^*(x), \ G_t(x,2) = 0, \ x \in \mathbb{R}^n, \\ G(y,t) = g(y,t), \ (y,t) \in S \times \mathbb{R}_+, \end{cases}$$

has a unique smooth solution $G(x,t)$ such that $G(x,2) = G_t(x,2) = 0$. Then, f^* vanishes up to infinite order on S.

- Domain of dependence
  - We have $G(x,t) = 0, \ K_+$, where $\{(x,t) \in B \times [0,2] : t - |x| \geq 1\}$
  - Also $G(x,t) = \mathcal{R}(f^*)(x,t), \ (x,t) \in K_-$, where $K_- = \{(x,t) \in B \times [0,2] : t + |x| \leq 1\}$.



- Hence: $\partial_x^\alpha \partial_t^k \mathcal{R}(f^*)(0,1) = 0$, for all α and k.
- For $f^*(x) = f_m(r) r^m Y_m(\theta)$. Repeat argument by Epstein-Kleiner (CPAM-'93) (for spherical mean transform on annular region):

$$\left[\left(\frac{d}{dr}\right)^k \prod_{s=1}^{m} \left(\frac{1}{n+(m-s)} r \frac{d}{dr} + 1 \right) \right] f_m(1) = 1, \ \forall k \geq 0.$$

PROOF of step 2

Summary:

$$\left[\left(\frac{d}{dr}\right)^k \prod_{s=1}^{m}\left(\frac{1}{n+(m-s)} r\frac{d}{dr}+1\right)\right] f_m(1) \quad = \quad 0, \ \forall k \geq 0,$$

$$\left(\frac{d^2}{dr^2}+\frac{n-1+2m}{r}\frac{d}{dr}\right)^k f_m(1) \quad = \quad 0, \ \forall k \geq 0.$$

Pick up the first m equations each:

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * & * \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & .. & 0 & 1 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \end{pmatrix} \begin{pmatrix} f_m(1) \\ f_m'(1) \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ f_m^{(2m-1)}(1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ 0 \end{pmatrix}.$$

The matrix is of full rank:

- The first m rows are linearly independent: obvious.
- The last m rows are linearly independent: by induction. Test with vector of the form $v_p = (\Psi_p(1), \Psi_p'(1), .., \Psi_p^{2m-1}(1))$, where $\Psi_p(t) = r^{-n-2p}$.

We obtain $f_m(1) = ... = f_m^{(2m-1)}(1) = 0$. Then $f_m^{(k)}(1) = 0$ for all $k \geq 0$.

SPHERICAL MEAN TRANSFORM ON $\mathbb{H}^n$

- Let $f \in C(\mathbb{H}^n)$, $\mathcal{R}(f)(x,t)$ is the mean value of f on the geodesics sphere $S(x,t)$:

$$\mathcal{R}(f)(x,t) = \frac{1}{|S(x,t)|} \int\limits_{S(x,t)} f(y) d\sigma(y).$$

- The function $G(x,t) = \mathcal{R}(f)(x,t)$ satisfies the Darboux equation:

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + (n-1)\coth(t)\frac{\partial G(x,t)}{\partial t} - \Delta G(x,0) = 0, \; (x,t) \in \mathbb{H}^n \times \mathbb{R}_+, \\ G(x,0) = f(x), \; G_t(x,0) = 0, \; x \in \mathbb{H}^n. \end{cases}$$

- Conversely, if G satisfies the above equation, $G = \mathcal{R}(f)$.
- $\mathcal{R}_S$ be the restriction of $\mathcal{R}$ to spheres centered at S.

Problem

Let S a sphere in $\mathbb{H}^n$ of radius R, and B the ball enclosed by S. Characterize the function g such that $g = \mathcal{R}_S(f)$ for some function $f \in C_0^\infty(\overline{B})$.

RANGE DESCRIPTION

Let

- h_λ satisfy:

$$\begin{cases} \left[\frac{d^2}{dt^2} + (n-1)\coth(r)\frac{d}{dt}\right] h_\lambda(r) = -\frac{(n-1)^2+4\lambda^2}{4} h_\lambda(t), \\ h_\lambda(0) = 1,\ h'_\lambda(0) = 0. \end{cases} \tag{2}$$

- φ_k is an eigenfunction of Dirichlet Laplacian in B:

$$-\Delta\varphi_k = \frac{(n-1)^2 + 4\lambda_k^2}{4}\varphi_k(x),\ \varphi_k|_S = 0.$$

Theorem (N.- arXiv:1107.1746v2)

Let g be a function defined on $S \times \overline{\mathbb{R}}_+$. Then, there is a function $f \in C_0^\infty(\overline{B})$ such that $g = \mathcal{R}_S(f)$ if and only if the following conditions are satisfied:

(a) **Smoothness & support condition:** $g \in C_0^\infty(S \times [0, 2R])$.

(b) **Orthogonality condition:**

$$\int_{\mathbb{R}_+}\int_S g(x,t)\partial_\nu\varphi_k(x)h_{\lambda_k}(t)\sinh^{n-1}(t)d\sigma(x)dt = 0.$$

PROOF

Restatement: Under the conditions (a) and (b), there is a global solution $G(x,t)$:

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + (n-1)\coth(t)\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, (x,t) \in \mathbb{H}^n \times \mathbb{R}_+, \\ G(x,0) = f(x),\ G_t(x,t) = 0,\ x \in \mathbb{H}^n, \\ G(y,t) = g(y,t),\ (y,t) \in S \times \mathbb{R}_+, \end{cases} \quad (3)$$

for some $f \in C_0^\infty(\overline{B})$.

1. Step 1: The equation

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   has a unique solution that satisfies $G(.,t) \to f^*$ for some function $f^* \in C^\infty(\overline{B})$.
   - Similar to the Euclidean case. Transform equation to wave equation. Tool: Paley-Wiener theory for the Fourier and Fourier-Legendre transforms.
2. Step 2: The function f^* vanishes up to infinite order at the boundary S of B.
3. Step 3: Let f be the zero extension of f^*. Then equation (3) has unique solution $G = \mathcal{R}(f)$. Hence, $g = \mathcal{R}_S(f)$.
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Theorem

Let $g \in C_0^\infty(S \times [0, 2R])$. Assume that

$$\begin{cases} \frac{\partial^2 G(x,t)}{\partial t^2} + \frac{n-1}{t}\frac{\partial G(x,t)}{\partial t} - \Delta G(x,t) = 0, \ x \in B, \\ G(x,0) = f^*(x), \ G_t(x,0) = 0, \ x \in B, \\ G(y,t) = g(y,t), \ (y,t) \in S \times \mathbb{R}_+, \end{cases}$$

has a unique smooth solution $G(x,t)$ such that $G(x,2R) = G_t(x,2R) = 0$. Then, f^* vanishes up to infinite order on S.

Assume $f^*(x) = f_m(r) Y_m(\theta)$. Show f_m vanishes up to infinite order at $r = R$.

- Iterating the equation k-times:

$$\Delta^k f(x) = \left(\frac{\partial^2}{\partial t^2} + (n-1)\coth(t)\frac{\partial}{\partial t}\right)^k g(x,0) = 0, \ \forall x \in S.$$

  This implies

$$\left(\frac{d^2}{dr^2} + (n-1)\coth(r)\frac{d}{dr} - \frac{m(m+n-2)}{\sinh^2(r)}\right)^k f_m(R) = 0, \ \forall k \geq 0.$$

PROOF of step 2

- Domain of dependence

$$G(x,t)=0,\ K_+ = \{(x,t)\in B\times[0,2R] : t - d_{\mathbb{H}^n}(x,0)\geq R\},$$
$$G(x,t)=\mathcal{R}(f^*)(x,t),\ K_- = \{(x,t)\in B\times[0,2R] : t + d_{\mathbb{H}^n}(x,0)\leq R\}.$$

- We take a different approach from Epstein-Kleiner's argument. The projection formulas are complicated.
- Recall $f^*(x)=f_m(r)Y_m(\theta)$. The function $G(x,t)=g_m(r,t)Y_m(\theta)$, where g_m satisfies the equation

$$\begin{cases} \left(\frac{\partial^2}{\partial t^2}+(n-1)\coth(t)\frac{\partial}{\partial t}\right)g_m(r,t)-\left(\frac{\partial^2}{\partial r^2}+(n-1)\coth(r)\frac{\partial}{\partial r}-\frac{m(m+n-2)}{\sinh^2(r)}\right)g_m(r,t)=0, \\ g_m(r,0)=f_m(r),\ 0\leq r\leq R, \quad g_m(0,t)=0,\ \forall t\geq R. \end{cases}$$

- The above equation can be symmetrized by $\mathcal{Q}_m=\prod_{s=1}^m\left(\frac{d}{dr}+(n+s-2)\coth(r)\right)$:

$$\left(\frac{\partial^2}{\partial t^2}+(n-1)\coth(t)\frac{\partial}{\partial t}\right)[\mathcal{Q}_m g_m](r,t)-\left(\frac{\partial^2}{\partial r^2}+(n-1)\coth(r)\frac{\partial}{\partial r}\right)[\mathcal{Q}_m g_m](r,t)=0$$

- Due to Helgason: $\mathcal{Q}_m f_m(r)=\mathcal{Q}_m g_m(r,0)=\mathcal{Q}_m g_m(0,r)=0,\ r\geq R$. Hence,

$$\left(\frac{d}{dr}\right)^k\prod_{s=1}^m\left(\frac{d}{dr}+(n+s-2)\coth(r)\right)f_m(R)=\left(\frac{d}{dr}\right)^k\mathcal{Q}_m f_m(R)=0.$$

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- The above equation can be symmetrized by $\mathcal{Q}_m = \prod_{s=1}^m \left(\frac{d}{dr} + (n+s-2)\coth(r)\right)$:

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- Due to Helgason: $\mathcal{Q}_m f_m(r) = \mathcal{Q}_m g_m(r,0) = \mathcal{Q}_m g_m(0,r) = 0, r \geq R$. Hence,

$$\left(\frac{d}{dr}\right)^k \prod_{s=1}^m \left(\frac{d}{dr} + (n+s-2)\coth(r)\right) f_m(R) = \left(\frac{d}{dr}\right)^k \mathcal{Q}_m f_m(R) = 0.$$

PROOF of step 2

Summary:

$$\left(\frac{d}{dr}\right)^k \prod_{s=1}^{m} \left(\frac{d}{dr} + (n+s-2)\coth(r)\right) f_m(R) = 0,\ k \geq 0,$$

$$\left(\frac{d^2}{dr^2} + (n-1)\coth(r)\frac{d}{dr} - \frac{m(m+n-2)}{\sinh^2(r)}\right)^k f_m(R) = 0,\ \forall k \geq 0.$$

Pick up the first m equations each:

$$\begin{pmatrix} 1 & * & * & * & * & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * & * & * & * \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & 0 & 1 & * & * & * & * & * & * \\ 0 & 0 & .. & 0 & 1 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ .. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \\ 0 & 0 & .. & 0 & 0 & * & * & * & * & * \end{pmatrix} \begin{pmatrix} f_m(R) \\ f'_m(R) \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ f_m^{(2m-1)}(R) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ * \\ * \\ * \\ * \\ * \\ * \\ * \\ 0 \end{pmatrix}.$$

Sophisticated algebra: $f_m(R) = \ldots = f_m^{(2m-1)}(R) = 0$. Then $f_m^{(k)}(R) = 0$ for all $k \geq 0$.

SPHERICAL MEAN TRANSFORM ON $\mathbb{S}^n$

- $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$. Injectivity radius of every point $x \in \mathbb{S}^n$ is π.
- Let $f \in C^\infty(\mathbb{S}^n)$, for each $x \in \mathbb{S}^n$ and $0 < t < \pi$, we define:

$$\mathcal{R}(f)(x,t) = \frac{1}{|S(x,t)|} \int\limits_{S(x,t)} f(y)d\sigma(y),$$

  where $S(x,t)$ sphere in $\mathbb{S}^n$ of radius t centered at x.
- $G(x,t) = \mathcal{R}(f)(x,t)$ satisfies the Darboux-type equation:

$$\begin{cases} \left[\frac{\partial^2}{\partial t^2} + (n-1)\cot(t)\frac{\partial}{\partial t} - \Delta\right] G(x,t) = 0, (x,t) \in \mathbb{S}^n \times (0,\pi), \\ G(x,0) = f(x), \; G_r(x,0) = 0, \; x \in \mathbb{S}^n. \end{cases}$$

- Conversely, if $G(x,t) \in C^\infty(\mathbb{S}^n \times [0,\pi])$ satisfies the above equation, then $G(x,t) = \mathcal{R}(f)(x,t)$.

Problem

Let $0 < R < \frac{\pi}{2}$ and $S = S(0,R)$. Characterize the range of $\mathcal{R}_S$.

RANGE DESCRIPTION FOR $\mathbb{S}^2$

For $n = 2$, let h_λ solves

$$\left[\frac{d^2}{dt} + \cot(t)\frac{d}{dt}\right] h(t) + \lambda(\lambda+1)h(t) = 0, \; h(0) = 1, \; h'(0) = 0.$$

Let φ_k be the eigenfuction of the Dirichlet Laplacian:

$$\begin{cases} -\Delta\varphi_k(x) = \lambda_k(\lambda_k+1)\varphi_k(x), \; x \in B, \\ \varphi_k|_S = 0. \end{cases}$$

Theorem

Let g be a function defined on $S \times [0,\pi]$. Then, there is a function $f \in C_0^\infty(\overline{B})$ such that $g = \mathcal{R}_S(f)$ if and only if g satisfies the following conditions:

(a) *Smoothness & support condition: $g \in C_0^\infty(S \times [0, 2R])$.*

(b) *Orthogonality condition:*

$$\int_0^\pi \int_S g(x,t)\partial_\nu\varphi_k(x)h_{\lambda_k}(t)\sin(t)d\sigma(x)dt = 0.$$

Remark

The same characterization also holds for $n > 2$.