Approximate reconstruction from circular mean data via classical summability

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Report of work done in collaboration with
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Notation:

$x = (x_1, x_2) \in \mathbb{R}^2$, also use y, ξ etc.

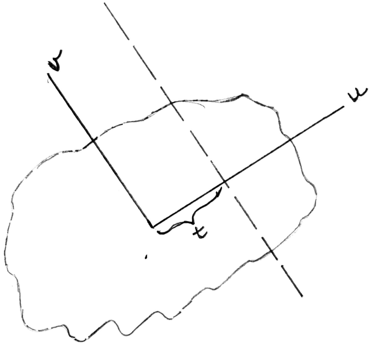
Scalar (inner, dot) product: $\langle x, y \rangle = x_1 y_1 + x_2 y_2$.

$u = u(\theta) = (\cos\theta, \sin\theta)$,

$v = u^\perp = u(\theta + \pi/2)$.

Radon transform:

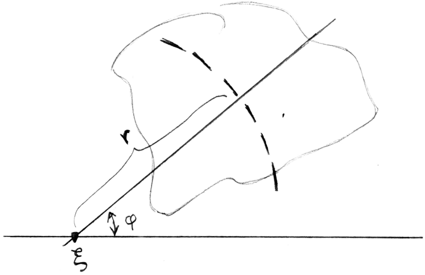
$$Rf(u,t) = \int_{-\infty}^{\infty} f(tu + sv)ds = f_u(t).$$



Applications are well known.

Circular mean transform:

$$\mathcal{M}f(\xi, r) = \int_0^{2\pi} f(\xi + ru(\phi))d\phi$$



Models the data acquisition scheme in photoacoustic, also known as thermoacoustic, tomography that is currently being tested for possible clinical applications.

Typical assumptions are that f has support in the unit disk and $\xi = u(\theta)$, $0 \leq \theta < 2\pi$.

We are interested in reconstruction of f in terms of the data $\mathcal{M}f(\xi, r)$, $\xi \in \Xi$, where Ξ is some appropriate collection of detectors exterior to the unit disk. An important special case is $\Xi = S^1$ the unit circle, the boundary of the unit disk with data $\mathcal{M}f(u(\theta), r)$, $0 \leq \theta < 2\pi$.

In the case $\Xi = S^1$ exact inversion formulas are known but, as in the case of the classical Radon transform, need to be regularized for numerical work.

Inversion formulas for f in terms of the data $\mathcal{M}f(u(\theta), r)$, $0 \leq \theta < 2\pi$, were first published in
FHR = D. Finch, M. Haltmeier, and Rakesh, Inversion of spherical means and the wave equation in even dimensions, *SIAM J. Appl. Math.* 68, no. 2, (2007), 392-412.

An alternate derivation and further generalizations can be found in
Y. A. Antipov, R, Estrada, and B. Rubin, Inversion formulas for spherical means in constant curvature spaces, (2011) preprint.

A different inversion formula can be found in
L. A. Kunyansky, Explicit inversion formulae for the spherical mean Radon transform, *Inv. Prob.* 23, (2007), 373-383.

Our method is a adaptation of a variant of a classical summability procedure used for Radon transform data outlined in
W. R. Madych, Summability and approximate reconstruction from Radon Transform data, *Contemporary Mathematics*, Vol 113 (1990), 189-219.

Recall the notion of a *ridge function*:

$$H(x) = h(\langle x, u \rangle)$$

If $f(x)$ has compact support

$$\begin{aligned} H * f(x) &= \int_{R^2} h(y) f(x-y) dy \\ &= \int_{-\infty}^{\infty} h(t) Rf(u, \langle x, u \rangle - t) dt \\ &= h * f_u(\langle x, u \rangle). \end{aligned}$$

If $K(x)$ is a *sum of ridge functions*, i. e.

$$K(x) = \int_0^{2\pi} h(\langle x, u(\theta)\rangle) \frac{d\theta}{2\pi}$$

then

$$\begin{aligned} K * f(x) &= \int_0^{2\pi} \left\{ \int_{-\infty}^{\infty} h(t) Rf(u(\theta), \langle x, u(\theta)\rangle - t) dt \right\} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} h * f_{u(\theta)}\big(\langle x, u(\theta)\rangle\big) \frac{d\theta}{2\pi}. \end{aligned} \tag{1}$$

If K is a an approximation of the identity then (1) gives rise to a reconstruction algorithm for f in terms of its Radon transform data.

Remark: There is a formula for h in terms of K that can be explicitly evaluated in certain cases.

Typical examples:

(2)
$$K(x) = \frac{1}{\pi}\begin{cases} 1 & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \frac{1}{\pi}\begin{cases} 1 & \text{if } |t| \leq 1 \\ 1 - |t|/(t^2-1)^{1/2} & \text{otherwise,} \end{cases}$$

(3)
$$K(x) = \frac{3}{\pi}\begin{cases} 1 - |x| & \text{if } |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \frac{3}{\pi}\begin{cases} 1 - \frac{\pi}{2}|t| & \text{if } |t| \leq 1 \\ 1 - t\arcsin(1/t) & \text{otherwise.} \end{cases}$$

$$(4) \qquad K(x) = \frac{1}{2\pi}\frac{1}{(1+|x|^2)^{3/2}} \qquad\qquad h(t) = \frac{1}{2\pi}\frac{1-t^2}{(1+t^2)^2},$$

Note that in all the above cases the family of functions parametrized by ϵ

$$K_\epsilon(x) = \frac{1}{\epsilon^2}K\left(\frac{x}{\epsilon}\right)$$

are well known approximations of the identity as $\epsilon \to 0$. The corresponding functions $h_\epsilon(t)$ of course, are given by

$$h_\epsilon(t) = \frac{1}{\epsilon^2}h\left(\frac{t}{\epsilon}\right)$$

We use the same philosophy to reconstruct f from its circular mean transform data.

G is *radial with center* ξ:

$$G(x) = g(|x - \xi|)$$

If f has compact support

$$\int_{\mathbb{R}^n} G(x)f(x)dx = \int_0^\infty \int_0^{2\pi} g(r)f(\xi + ru(\theta))rd\theta\, dr$$
$$= \int_0^\infty g(r)\mathcal{M}f(\xi, r)rdr.$$

If $K(x,y)$ is a *sum of radial functions* in the variable y, i. e.

$$K(x,y) = \int_{\Xi} k(x,\xi,|y-\xi|)d\mu(\xi)$$

then

$$(5) \quad \int_{\mathbb{R}^2} K(x,y)f(y)dy = \int_{\Xi} \Big\{ \int_0^{\infty} k(x,\xi,r)\mathcal{M}f(\xi,r)rdr \Big\} d\mu(\xi).$$

If f is sufficiently regular and has support in a region Ω and $K(x,y)$ is a good approximation of the identity in y at each $x \in \Omega$ then then identity (5) represents an approximate reconstruction of f in terms of the data $\mathcal{M}f(\xi,r)$, $\xi \in \Xi$ and $r > 0$.

Such a kernel $K(x, y)$ can be conveniently viewed as a family of functions in the y variable parameterized by x, each member of which is a sum of radial functions with centers in Ξ.

As alluded to earlier, we will study the case $\Xi = S^1 = \{\xi = u(\theta) : 0 \leq \theta < 2\pi\}$ with $d\mu(\xi) = \frac{d\theta}{2\pi}$ and f supported in $B = \{x : |x| < 1\}$.

Remark: I don't know how to solve

$$G(y) = \int_{S^1} g(|y - u|) d\mu(u)$$

for g and μ in terms of G.

This leaves us with the problem of how to construct such $K(x, y)$ and $k(x, \xi, |y - \xi|)$ pairs?

Note that

$$\lim_{r\to\infty}\left\{|x-ru|-|y-ru|\right\}=\langle y-x,u\rangle.$$

This suggests that, roughly speaking, if the detector $\xi = ru$ is relatively far from x and y then $|x-\xi|-|y-\xi|$ looks like $\langle y-x,u\rangle$.

We know that

$$\epsilon^{-2}K((y-x)/\epsilon)=\int_0^{2\pi}\frac{1}{\epsilon^2}h\Big(\frac{\langle y-x,u(\theta)\rangle}{\epsilon}\Big)\frac{d\theta}{2\pi}$$

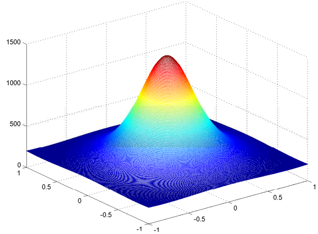
is a good approximation of the identity at x with an appropriate choice of h. i. e. one of the examples of K, h pairs.

Hence, it is not unreasonable to expect that

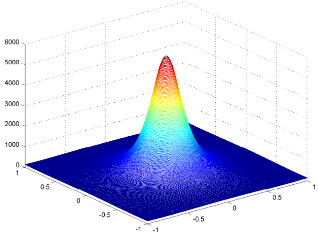
$$K_1(x,y;\epsilon)=\int_0^{2\pi}\frac{1}{\epsilon^2}h\Big(\frac{|x-u(\theta)|-|y-u(\theta)|}{\epsilon}\Big)\frac{d\theta}{2\pi}$$

where h comes from the ridge function representation of a kernel K, i. e. one of the examples (3) or (4), looks like a summability kernel or approximate identity at x. At least for x and y close to the origin.

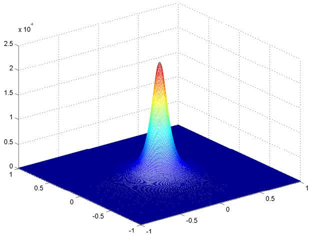
Plots of $K_1(x, y; \epsilon)$ for fixed x and ϵ as a function of y. Here $h(t) = \frac{1}{2\pi} \frac{1-t^2}{(1+t^2)^2}$ as in example (4).



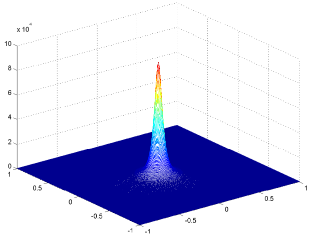
$x = (0, 0)$, $\epsilon = 0.5$.



$x = (0, 0)$, $\epsilon = 0.25$.

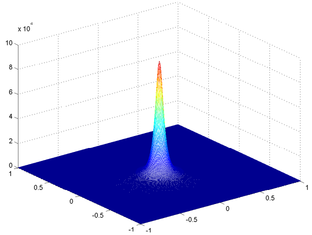


$x = (0, 0)$, $\epsilon = 0.125$.

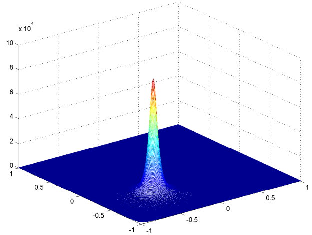


$x = (0, 0)$, $\epsilon = 0.0625$.

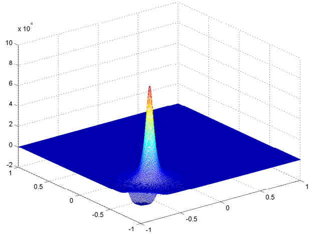
More plots of $K_1(x, y; \epsilon)$ for fixed x and ϵ as a function of y with the same $h(t)$.



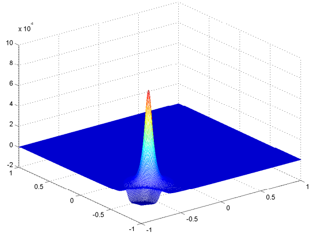
$x = (0, 0)$, $\epsilon = 0.0625$.



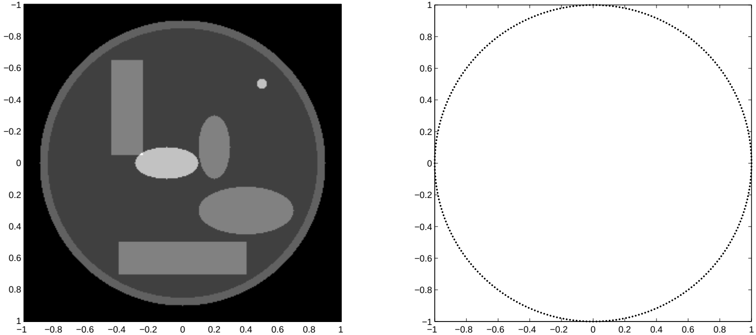
$x = -\frac{1}{2\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.



$x = -\frac{3}{4\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.



$x = -\frac{7}{8\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.



Phantom and detectors.

We use a discretization of

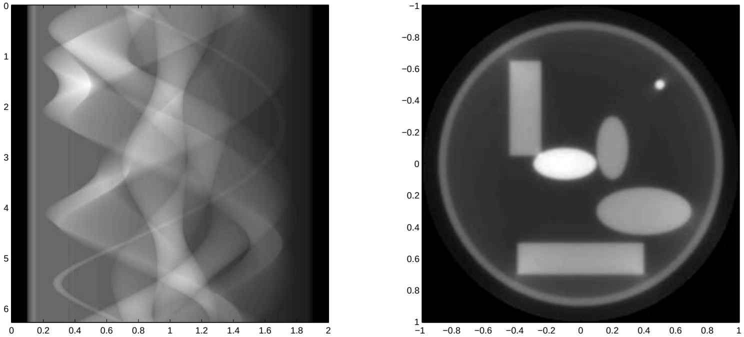
$$\int_{|y|<1} K_1(x,y;\epsilon)f(y)dy$$

$$= \int_0^{2\pi} \left\{ \int_0^2 h_\epsilon(|x-u(\theta)|-r)\mathcal{M}f(u(\theta),r)rdr \right\} \frac{d\theta}{2\pi}.$$

Reconstruction

$$\tilde{f}(x) = \frac{C}{MN} \sum_{j=1}^{N} \left\{ \sum_{i=1}^{M} h_\epsilon(|x - u_{\theta_j}| - r_i) \mathcal{M} f(u_{\theta_j}, r_i) r_i \right\}$$

$$h_\epsilon(t) = \frac{\epsilon^2 - t^2}{(\epsilon^2 + t^2)^2}, \quad \epsilon = 0.01, \quad M = 299, \quad N = 300.$$



Data and reconstruction.

These and similar numerical experiments suggest that $K_1(x, y; \epsilon)$ is a summability kernel and a good approximation of the identity at x for $|x| < 1$ as a function of y, $|y| < 1$, for sufficiently small ϵ.

Further numerical experiments suggest that the set of detectors Ξ need not be restricted to circles. For example

$$K(x, y; \epsilon) = C \sum_{\xi_i \in \Xi} h_\epsilon(|x - \xi_i| - |y + \xi_i|)$$

will still be a good approximation of the identity at x for $|x| < 1$ as a function of y, $|y| < 1$, for appropriate ϵ as long as, roughly speaking, the set of detectors Ξ is a sufficiently dense set surrounding the unit disk. F. Filbir, R. Hielscher, and W.R. Madych, Reconstruction from circular and spherical mean data, *Applied and Computational Harmonic Analysis* 29, (2010), 111-120.

Is it true that

$$\lim_{\epsilon \to 0} \int_{|y|<1} K_1(x, y; \epsilon) f(y) dy = f(x)$$

whenever f is bounded, vanishes outside the unit disk, and is continuous at x?

To hopefully simplify the matter try working with

$$K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\Big(\frac{|x - u(\theta)|^2 - |y - u(\theta)|^2}{2\epsilon}\Big) \frac{d\theta}{2\pi}$$

which is also a sum of radial functions in the variable y, seems pretty much like $K_1(x, y; \epsilon)$, but the argument $|x - u(\theta)|^2 - |y - u(\theta)|^2$ is algebraically easier to work with.

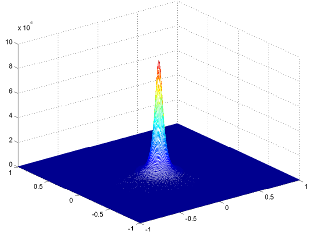
Note that $K_2(x, y; \epsilon)$ can be re-expressed as

$$K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\Big(\Big\langle \frac{x-y}{\epsilon}, u(\theta) - \frac{x+y}{2} \Big\rangle\Big) \frac{d\theta}{2\pi}$$

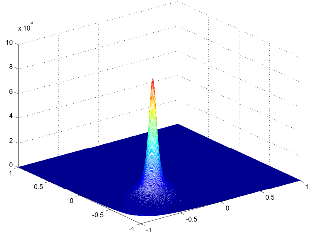
or

$$K_2(x, y; \epsilon) = \int_0^{2\pi} \frac{1}{\epsilon^2} h\Big(\Big\langle \frac{x-y}{\epsilon}, u(\theta) \Big\rangle + \frac{|y|^2 - |x|^2}{2}\Big) \frac{d\theta}{2\pi}.$$

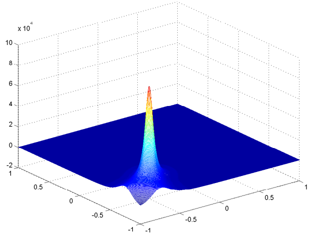
Plots of $K_2(x, y; \epsilon)$ for fixed x and ϵ as a function of y with $h(t)$ as in (4).



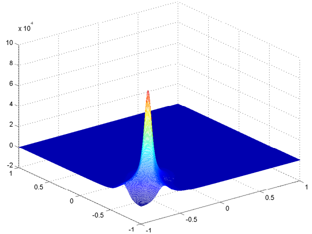
$x = (0, 0)$, $\epsilon = 0.0625$.



$x = -\frac{1}{2\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.

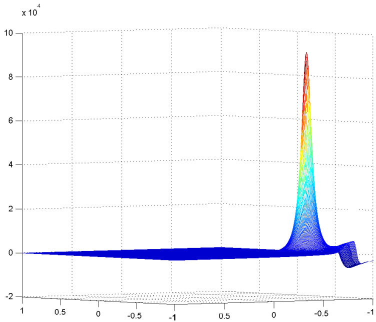


$x = \ldots 3 \ldots (1, 1)$, $\epsilon = 0.0625$

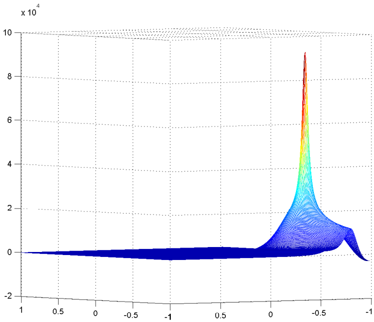


$x = \ldots 7 \ldots (1, 1)$, $\epsilon = 0.0625$

Comparison of the plots of $K_1(x, y; \epsilon)$ and $K_2(x, y; \epsilon)$ for fixed x and ϵ as a functions of y.



K_1 at $x = -\frac{7}{8\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.



K_2 at $x = -\frac{7}{8\sqrt{2}}(1, 1)$, $\epsilon = 0.0625$.

Theorem: If h is the function in example (4), that is

$$h(t) = \frac{1}{2\pi} \frac{1 - t^2}{(1 + t^2)^2},$$

then

$$\lim_{\epsilon \to 0} \int_{|y|<1} K_2(x, y; \epsilon) f(y) dy = c(x) f(x)$$

where

$$c(x) = \frac{\pi}{1 - |x|^2}$$

whenever f is bounded, vanishes outside the unit disk, and is continuous at x.

This is a corollary of the following:

Lemma: If h is the function in example (4) then

$$\left| \frac{1}{2\pi} \int_0^{2\pi} h(\langle z, u(\theta) - x \rangle) d\theta \right| \leq \frac{C}{1 + |z|^3}$$

for all $z \in \mathbb{R}^2$ where C is a constant that depends only on x when $|x| < 1$.

Proof of Lemma:
Use residues to to evaluate the integral and get

$$\int_0^{2\pi} h(\langle z, u(\theta) - x\rangle)d\theta = c\,\mathrm{Re}\,\frac{\langle z, x\rangle + i}{\left((\langle z, x\rangle + i)^2 - |z|^2\right)^{3/2}}.$$

Follow this by several pages of algebraic manipulations together with applications of appropriate inequalities to get the desired result.

Note 1: The analytic nature of $h(t)$ and the change to $|x - \xi|^2 - |y - \xi|^2$ allowed us to do this.

Note 2: Want argument which depends only on the integrability of

$$K(x) = \int_0^{2\pi} h(\langle x, u(\theta)\rangle)d\theta$$

over R^2.

Corollary 1 of Theorem: If x is in the unit disk B, and f is in $C^2(B)$ then

$$(6)\quad \frac{f(x)}{(1-|x|^2)}$$
$$= \frac{-1}{\pi^2}\int_0^{2\pi}\left\{\frac{1}{2}\int_0^{\sqrt{2}a}\left\{M(r)+M\left(\sqrt{2a^2-r^2}\right)-2M(a)\right\}\frac{r}{\left(r^2-a^2\right)^2}dr\right.$$
$$\left.+\int_{\sqrt{2}a}^{\infty}M(r)\frac{r}{\left(r^2-a^2\right)^2}dr-\frac{M(a)}{a^2}\right\}d\theta$$

where $M(r) = \mathcal{M}f(\xi, r)$ and $a = |x - u(\theta)|$.

Remark: f in $C^2(B)$ is overkill. I suspect that f in $L^p(B)$ for $p \geq p_0$ is sufficient.

Corollary 2 of Theorem: If Ω is the unit disk, x is in Ω, and f is in $C^2(\Omega)$ then $f(x) =$

$$\frac{1-|x|^2}{2\pi}\int_0^{2\pi}\left\{\int_0^2 \log\left(\left|r^2-|x-u(\theta)|^2\right|\right)\frac{d}{dr}\left(\frac{1}{2r}\frac{d\mathcal{M}f(u(\theta),r)}{dr}\right)dr\right\}d\theta.$$

Remark 1: Analogues of the Theorem and Corollaries are valid when the unit disk is replaced by a more general elliptical region and the set of detectors Ξ, currently a circle, is replaced by a more general ellipse. In this case $d\mu(\xi)$ is not the usual arclength.

Remark 2: The inversion formula of the Corollary should be compared with the inversion formula in FHR.

$$f(x) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^2 \log\left(\left|r^2-|x-u(\theta)|^2\right|\right)\frac{d}{dr}\left(r\,\frac{d\mathcal{M}f(u(\theta),r)}{dr}\right)dr\,d\theta.$$

Analogous results are valid for spherical mean transforms in higher dimensions n.

In fact, in the case $n = 3$ a stronger version of the Theorem is valid. The function h need not be analytic. It suffices that

$$K(x) = \frac{1}{4\pi} \int_{S^2} h(\langle x, u \rangle) d\sigma(u)$$

be integrable over $\mathbb{R}^3$.