

# A CLASS OF SINGULAR FOURIER INTEGRAL OPERATORS IN SYNTHETIC APERTURE RADAR IMAGING

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## WHAT IS SAR IMAGING?

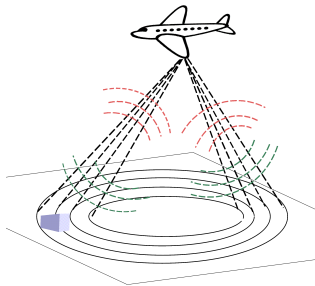
Region of interest illuminated by electromagnetic (EM) waves from a moving airborne platform.

For each fixed antenna position, EM waves are sent for a time interval and the scattered waves measured.

Region imaged based on the measurement of scattered waves.

In monostatic SAR, one airborne platform has the transmitter and receiver.

In bistatic SAR, the transmitter and receiver are on independently moving airborne platforms.

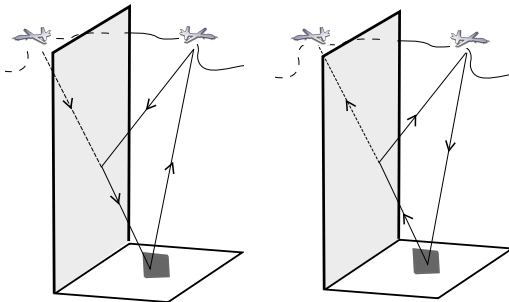


# COMMON MIDPOINT ACQUISITION GEOMETRY

Focus of this talk: Common midpoint acquisition geometry – transmitter and receiver used in imaging move at equal speeds away from a common midpoint along a straight line.

Geometry arises in bistatic imaging.

Geometry also arises in certain multiple scattering scenarios.



Quinto and I in an earlier work considered a common offset acquisition geometry in the context of SAR:

The adjoint operator  $F^*$  introduces artifacts.

We analyzed the normal operator  $F^*F$ . We showed that  $F^*F$  belongs to the class of cleanly intersecting Lagrangians introduced by Melrose-Uhlmann and Guillemin-Uhlmann.

An important consequence: The strength of the artifacts is the same as that of the true singularities.

Ambartsoumian, Quinto and I considered 2-dimensional bistatic imaging problem in ultrasound imaging, where the emitter and receiver move in a circular trajectory at a constant distance apart.

Here again, the adjoint operator introduces artifacts.

The analysis of  $F^*F$  in general is a difficult problem. However, we showed that  $F^*F$  restricted to support inside the circular trajectory is an elliptic pseudodifferential operator.

Quinto will talk about this as well show image reconstructions.

First analyze the linearized forward scattering operator  $F$ . This a Fourier integral operator.

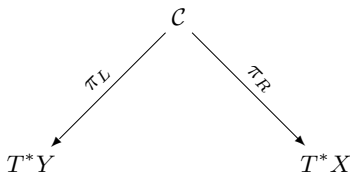
Next analyze the composition of  $F$  with its  $L^2$  adjoint  $F^*$ .

One of the main goals: Understand the distribution class of  $F^*F$ .

Well known that composition of two FIOs is not an FIO.

Two geometric conditions where the composition is an FIO: Transverse intersection of Hörmander and clean intersection of Duistermaat - Guillemin, and Weinstein.

In general, when these geometric conditions do not hold, we analyze the mapping properties of the canonical relation of  $F$ . Let  $F : \mathcal{E}'(X) \rightarrow \mathcal{D}'(Y)$  be an FIO and let  $\mathcal{C} \subset (T^*Y \times T^*X) \setminus \{0\}$  be the canonical relation associated to  $F$ . Analyze the mapping properties of the projection maps:



Assume the transmitter (T) and receiver (R) are at the same height  $h$  above the ground and move in opposite directions at equal speeds. Let  $\gamma_T(s) = (s, 0, h)$  and  $\gamma_R(s) = (-s, 0, h)$  for  $s \in (0, \infty)$ .

The linearized scattering operator we use in this talk is

$$FV(s, t) = \int e^{-i\omega(t - \frac{1}{c_0}R(s, x))} a(s, x, \omega) V(x) dx d\omega \text{ for } (s, t) \in (0, \infty) \times (0, \infty).$$

Here

$$R(s, x) = |x - \gamma_T(s)| + |x - \gamma_R(s)|$$

is the bistatic distance. We assume  $a$  satisfies an amplitude estimate. That is  $a \in S^{m+(1/2)}$  with  $\omega$  as the frequency variable.

Denote the  $\{(s, t)\}$  space as  $Y$  and the plane  $\{(x_1, x_2)\}$  as  $X$  from now on.

For the composition of  $F$  with its adjoint to be well-defined, we multiply  $a$  by a smooth cut-off function identically 1 and supported in a compact subset of  $Y$ .

Our imaging method cannot image a neighborhood of the common midpoint (that is the origin). Modify  $a$  further by multiplying by a smooth cut-off function such that it is 0 in a neighborhood of  $(s, t) : |t - 2\sqrt{s^2 + h^2}| < \varepsilon$ .

We have

$$FV(s, t) = \int e^{-i\omega(t - \sqrt{(x_1-s)^2 + x_2^2 + h^2} - \sqrt{(x_1+s)^2 + x_2^2 + h^2})} a(s, t, x, \omega) V(x) dx d\omega.$$



## THEOREM

- $F$  is an FIO of order  $m$
- The canonical relation  $\mathcal{C}$  associated to  $F$  is given by

$$\mathcal{C} = \left\{ \left( s, t, -\omega \left( \frac{x_1 - s}{\sqrt{(x_1 - s)^2 + x_2^2 + h^2}} - \frac{x_1 + s}{\sqrt{(x_1 + s)^2 + x_2^2 + h^2}} \right), -\omega; \right. \right. \\ \left. \left. x_1, x_2, -\omega \left( \frac{x_1 - s}{\sqrt{(x_1 - s)^2 + x_2^2 + h^2}} + \frac{x_1 + s}{\sqrt{(x_1 + s)^2 + x_2^2 + h^2}} \right), \right. \right. \\ \left. \left. -\omega \left( \frac{x_2}{\sqrt{(x_1 - s)^2 + x_2^2 + h^2}} + \frac{x_2}{\sqrt{(x_1 + s)^2 + x_2^2 + h^2}} \right) \right) \text{ where} \right. \\ \left. s > 0, t = \sqrt{(x_1 - s)^2 + x_2^2 + h^2} + \sqrt{(x_1 + s)^2 + x_2^2 + h^2}, x \neq 0, \text{ and } \omega \neq 0 \right\},$$

and  $\mathcal{C}$  has global parameterization

$$(0, \infty) \times (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \ni (s, x_1, x_2, \omega) \rightarrow \mathcal{C}.$$

- Let  $\pi_L : \mathcal{C} \rightarrow T^*Y$  and  $\pi_R : \mathcal{C} \rightarrow T^*X$  be the left and right projections respectively. Then  $\pi_L$  and  $\pi_R$  drop rank simply by one on a set  $\Sigma = \Sigma_1 \cup \Sigma_2$  where in the coordinates  $(s, x, \omega)$ ,  
 $\Sigma_1 = \{(s, x_1, 0, \omega) | s > 0, |x_1| > \epsilon', \omega \neq 0\}$  and  
 $\Sigma_2 = \{(s, 0, x_2, \omega) | s > 0, |x_2| > \epsilon', \omega \neq 0\}$  for  $0 < \epsilon'$  small enough.
- $\pi_L$  ( $\pi_R$ ) has a fold (blowdown) singularity along  $\Sigma$ .

We consider the normal operator  $F^*F$  and show that it can be decomposed as of sum of distributions each belonging to an  $I^{p,l}$  class associated to two cleanly intersecting Lagrangians.

## THEOREM

*Then  $F^*F$  can be decomposed into a sum belonging to  $I^{2m,0}(\Delta, C_1) + I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3) + I^{2m,0}(C_2, C_3)$ .*

The Lagrangians  $\Delta$  and  $C_i$  for  $i = 1, 2$  and  $3$  will be defined later.

We have

$$\det((\pi_L)_*) = \frac{4x_1x_2s\omega}{A^2B^2} \left(1 + \frac{(x_1^2 - s^2 + x_2^2 + h^2)}{AB}\right),$$

where  $A = \sqrt{(x_1 - s)^2 + x_2^2 + h^2}$  and  $B = \sqrt{(x_1 + s)^2 + x_2^2 + h^2}$ . The projection drops ranks along  $\Sigma_1 = \{(s, x_1, 0, \omega), x_1 \neq 0\}$  and  $\Sigma_2 = \{(s, 0, x_2, \omega), x_2 \neq 0\}$  and that the determinant vanishes exactly to first order.

Easy to show that the left projection  $\pi_L$  is a fold along the union of the axes and that the right projection is a blowdown along the union of the axes.

Locally  $\pi_L$  is of the form  $\pi_L(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3, z_4^2/2)$ , and locally  $\pi_R$  is of the form,  $\pi_R(z_1, z_2, z_3, z_4) = (z_1, z_2, z_3z_4, z_4)$ .

Note that  $C$  is even with respect to both  $x_1$  and  $x_2$ . In other words  $C$  is a 4-1 relation. This suggests that  $\pi_L$  (respectively  $\pi_R$ ) has two fold (respectively blowdown) sets.

We have

$$F^* FV(x) = \int e^{i\omega(t-(|x-\gamma_T(s)|+|x-\gamma_R(s)|))-\tilde{\omega}(t-(|y-\gamma_T(s)|+|y-\gamma_R(s)|))} \\ \times \overline{a(s,t,x,\omega)} a(s,t,y,\tilde{\omega}) V(y) ds dt d\omega d\tilde{\omega} dy.$$

After applications of stationary phase method, the Schwartz kernel of this operator is

$$K(x,y) = \int e^{i\Phi(x,y,s,\omega)} \tilde{a}(x,y,s,\omega) ds d\omega,$$

where

$$\Phi = \omega (|y - \gamma_T(s)| + |y - \gamma_R(s)| - (|x - \gamma_T(s)| + |x - \gamma_R(s)|)).$$

Using Hörmander-Sato Lemma, we have

$$WF(K)' \subset \Delta \cup C_1 \cup C_2 \cup C_3,$$

where  $\Delta$  is the diagonal in  $T^*X \times T^*X$  and the Lagrangians  $C_i$  for  $i = 1, 2, 3$  are the graphs of the following functions  $\chi_i$  for  $i = 1, 2, 3$  on  $T^*X$ :

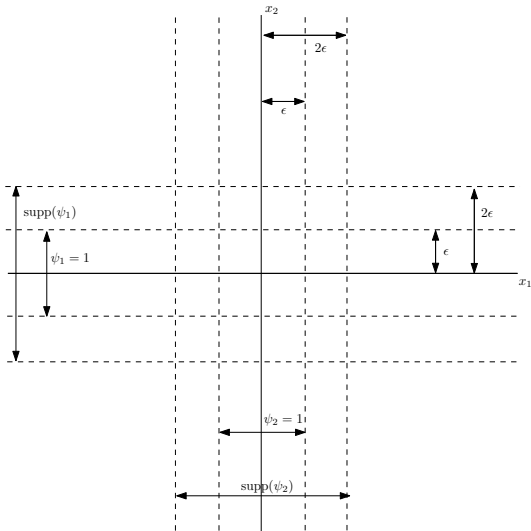
$$\chi_1(x, \xi) = (x_1, -x_2, \xi_1, -\xi_2), \chi_2(x, \xi) = (-x_1, x_2, -\xi_1, \xi_2) \text{ and } \chi_3 = \chi_1 \circ \chi_2.$$

Also it is easy to see that

$\Delta$  and  $C_1$ ,  $\Delta$  and  $C_2$ ,  $C_1$  and  $C_3$ ,  $C_2$  and  $C_3$  intersect cleanly in codimension 2,  $\Delta \cap C_3 = C_1 \cap C_2 = \emptyset$

# PROOF SKETCH

Since the projection maps drop rank along  $\Sigma_1 \cup \Sigma_2$ , we decompose  $F$  such that the canonical relation of  $F$  is supported either near these sets or away from it.



Then we write  $F = F_0 + F_1 + F_2 + F_3$  where  $F_i$  are given in terms of their kernels

$$K_{F_0} = \int e^{-i\varphi} a \psi_1 \psi_2 d\omega, \quad K_{F_1} = \int e^{-i\varphi} a \psi_1 (1 - \psi_2) d\omega,$$

$$K_{F_2} = \int e^{-i\varphi} a (1 - \psi_1) \psi_2 d\omega, \quad K_{F_3} = \int e^{-i\varphi} a (1 - \psi_1) (1 - \psi_2) d\omega,$$

where  $\varphi$  is the phase function of  $F$ :

$$\varphi = \omega(t - \sqrt{(x_1 - s)^2 + x_2^2 + h^2} - \sqrt{(x_1 + s)^2 + x_2^2 + h^2}).$$

Using the decomposition of  $F$ ,  $F^* F$  can be written as

$$F^* F = F_0^* F + (F_1 + F_2)^* F_0 + F_1^* F_1 + F_2^* F_2 + F_1^* F_2 + F_2^* F_1 + F_1^* F_3 + F_2^* F_3 + F_3^* F$$

We analyze each of these terms.

$F_0, F_1^* F_2$  and  $F_2^* F_1$  are smoothing.

$F_1^* F_3, F_2^* F_3$  and  $F_3^* F$  can be decomposed as a sum of operators belonging to the space  $I^{2m}(\Delta) + I^{2m}(C_1 \setminus \Delta) + I^{2m}(C_2 \setminus \Delta) + I^{2m}(C_3 \setminus (C_1 \cup C_2))$ . This is because each of these compositions is covered by the transverse intersection calculus

Now we are left with the terms  $F_1^* F_1$  and  $F_2^* F_2$ .

## THEOREM

- ①  $F_1^* F_1 \in I^{2m,0}(\Delta, C_1) + I^{2m,0}(C_2, C_3)$ .
- ②  $F_2^* F_2 \in I^{2m,0}(\Delta, C_2) + I^{2m,0}(C_1, C_3)$ .

Decompose  $F_1$  as

$$F_1 = F_1^+ + F_1^-,$$

Now

$$F_1^* F_1 = (F_1^+)^* F_1^+ + (F_1^-)^* F_1^- + (F_1^+)^* F_1^- + (F_1^-)^* F_1^+.$$

We now use the iterated regularity theorem of Greenleaf-Uhlmann: If  $u \in \mathcal{D}'(X \times Y)$  then  $u \in I^{p,l}(\Lambda_0, \Lambda_1)$  if there is an  $s_0 \in \mathbb{R}$  such that for all first order pseudodifferential operators  $P_i$  with principal symbols vanishing on  $\Lambda_0 \cup \Lambda_1$ , we have  $P_1 P_2 \dots P_r u \in H_{loc}^{s_0}$ .

Using this result we can show that,

$$(F_1^+)^* F_1^+, (F_1^-)^* F_1^- \in I^{2m,0}(\Delta, C_1).$$

and

$$(F_1^-)^* F_1^+, (F_1^+)^* F_1^- \in I^{2m,0}(C_2, C_3).$$



Using the properties of the  $I^{p,l}$  classes,  $F^*F \in I^{2m,0}(\Delta, C_1)$  implies that  $F^*F \in I^{2m}(\Delta \setminus C_1)$  and  $F^*F \in I^{2m}(C_1 \setminus \Delta)$ . This means that  $F^*F$  has the same order on both  $\Delta$  and  $C_1$  which implies that the artifact  $C_1$  has the same strength as the initial singularities given by  $\Delta$ . Similarly for  $C_2$  and  $C_3$ .

Note that  $C_1$  gives an artifact that is a reflection in the  $x_1$  axis,  $C_2$  gives an artifact that is a reflection in the  $x_2$  axis, and  $C_3$  gives an artifact that is a reflection in the origin.

To deal with distributions associated to more than two cleanly intersecting Lagrangians, perhaps we require an extension of  $I^{p,l}$  classes to, say,  $I^{p,l,m,n}$  classes. Once these classes are well defined and the properties established, the distribution encountered in this SAR problem could fit into that framework.