Inverse scattering on generalized arithmetic surfaces

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Geometric analysis on
Euclidean and homogeneous spaces
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I am going to talk about our joint work (put in arXiv recently)

Conic singularities, generalized scattering matrix, and inverse scattering on asymptotically hyperbolic surfaces, H. Isozaki, Y. Kurylev and M. Lassas.

First let us recall the following theorem, which, although not used directly, will help understanding the basic idea.

Theorem (S. Helgason)

Any solution of the Helmholtz equation $-\Delta_g u = \lambda u$ on the Poincaré disc is written by the Poisson integral

$$u = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1-|z|^2}{|e^{\sqrt{-1}\theta}-z|}\right)^s f(\theta)d\theta,$$

where $f(\theta)$ is Sato's hyperfunction on the boundary.

This theorem is extended by

Kashiwara, Kowata, Minemura, Okamoto, Oshima and Tanaka, Ann. of Math. 107 (1978), 1-39

to the general symmetric space.

Contents

§1 Scattering theory on the hyperbolic spaces

We recall basic facts from scattring theory and 2-dim. hyperbolic manifolds.

Euclidean case

The soultion of the Schrödinger equation on R^n

$$Hu = (-\Delta + V(x))u = \lambda u, \quad \lambda > 0$$

in a suitable class can be written as follows

$$u = \mathcal{F}_0(\lambda)^*\phi - R(\lambda + i0)V\mathcal{F}_0(\lambda)^*\phi,$$

$$\phi \in L^2(S^{n-1}),$$

$$\mathcal{F}_0(\lambda)^*\phi = (2\pi)^{-n/2}\int_{S^{n-1}} e^{i\sqrt{\lambda}\omega\cdot x}\phi(\omega)d\omega.$$

This is called the Herglotz or Poisson integral. Here

$$R(\lambda \pm i0) = \lim_{\epsilon\to 0}(H - \lambda \mp i\epsilon)^{-1}$$

and this limit is known to exist between suitable Banach spaces (rigged Hilbert space)

$$R(\lambda \pm i0) : \mathcal{B} \to \mathcal{B}^*,$$

$$\mathcal{B} \subset L^2(\mathrm{R}^n) \subset \mathcal{B}^*.$$

What we learn from this formula?

- **There is a *space at infinity*, i.e.** S^{n-1}.
- **There is an *integral formula at infinity* by which all solutions of the Schrödinger equation in a *suitable class* can be represented.**

What is S-matrix?

By applying the stationary phase method on the sphere, one can see that

$$\forall \phi_{in} \in L^2(S^{n-1}), \quad \exists 1 u, \quad \exists 1 \phi_{out} \in L^2(S^{n-1}), \quad s.t.$$

$$(-\Delta + V - \lambda)u = 0,$$

$$u \simeq C_-(\lambda)\frac{e^{-i\sqrt{\lambda}r}}{r^{(n-1)/2}}\phi_{in}(\omega) + C_+(\lambda)\frac{e^{i\sqrt{\lambda}r}}{r^{(n-1)/2}}\phi_{out}(\omega)$$

as $r = |x| \to \infty$**.**

The mapping

$$S(\lambda) : L^2(S^{n-1}) \ni \phi_{in} \to \phi_{out} \in L^2(S^{n-1})$$

is unitary, and is called the *(geometric) S-matrix***.**

General belief

In a suitable setting,

S-matrix determines the original physical system.

This will be true not only for potentials but also perturbations by metrics.

Besov type spaces

There is a long history for the *rigged Hilbert space*, but the following Besov type space introduced by Agmon-Hörmander (1976) is now regarded as the most appropriate one

$$\mathcal{B} \subset L^2(\mathrm{R}^n) \subset \mathcal{B}^*,$$

$$u \in \mathcal{B}^* \Longleftrightarrow \sup_{R>1} \frac{1}{R} \int_{|x|<R} |u(x)|^2 dx < \infty.$$

Note that

$$\frac{e^{\pm i\sqrt{\lambda}r}}{r^{(n-1)/2}} \in \mathcal{B}^*.$$

Hyperbolic space case

The action of $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathrm{R})$ on C_+ :

$$SL(2, \mathrm{R}) \times \mathrm{C}_+ \ni (\gamma, z) \to \gamma \cdot z = \frac{az + b}{cz + d} \in \mathrm{C}_+.$$

For a discrete subgroup (Fuchsian group) $\Gamma \subset SL(2, \mathrm{R})$, consider the fundamental domain

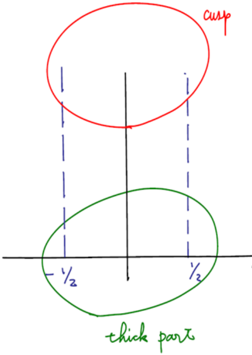
$$\mathcal{M}_\Gamma = \Gamma \backslash \mathrm{H}^2$$

Example 1. Translation

$$\Gamma : z \longrightarrow z + 1$$

$$\mathcal{M}_\Gamma = \Big(-\frac{1}{2}, \frac{1}{2}\Big) \times (0, \infty),$$

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}$$



Example Γ corresponds to the horizontal translation

$$z \to \gamma \cdot z = z + 1.$$

The resulting space is the cylinder

$$\mathcal{M}_{\Gamma} = \Big(-\frac{1}{2}, \frac{1}{2} \Big] \times (0, \infty).$$

It has two infinities :

- Around $y = 0$, the infinity of infinite volume, which we call the *regular infinity*.
- Around $y = \infty$, the infinity of finite volume, which we call the *cusp*.

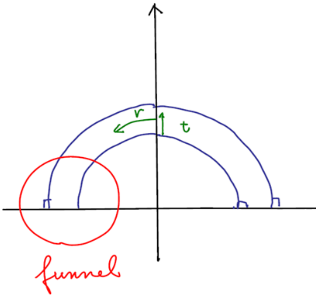
Example 2. Dilation

$$\Gamma : z \to \lambda z \quad (\lambda > 1)$$

$$ds^2 = (dr)^2 + \cosh^2 r (dt)^2$$

$$(y = 2e^{-r})$$

$$= \left(\frac{dy}{y}\right)^2 + \left(\frac{1}{y} + \frac{y}{4}\right)^2 (dt)^2$$



Classification of the 2-dim. hyperbolic spaces

Γ (or $\mathcal{M}_\Gamma$) is said to be *geomerically finite*
$\Longleftrightarrow$ $\mathcal{M}_\Gamma$ is a finite sided convex domain.
$\Longleftrightarrow$ Γ is finitely generated.
$\Longrightarrow$ $\exists$ a compact set $\mathcal{K} \subset \mathcal{M}_\Gamma$ s.t. $\mathcal{M}_\Gamma \subset \mathcal{K}$ consists of a finite number of cusps and the funnel.

Fundamental domains for the Fuchsian group of the 1^{st} kind

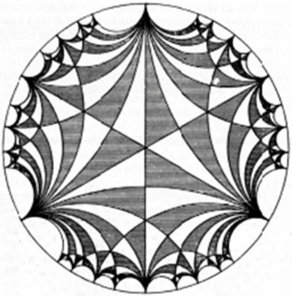


Figure 3.17. Another tessellation of the unit disc. (From Klein and Fricke [1]. Reprinted by permission of Teubner.)

A Fuchsian group Γ is of the 1st kind (we omit its definition)

$\iff \mathcal{M}_\Gamma$ is of finite volume.

$\Longrightarrow \mathcal{M}_\Gamma$ is geometrically finite.

If Γ is of the 1st kind, the ends of $\mathcal{M}_\Gamma$ consists only of cusps.

Usually, one compactifies $\mathcal{M}_\Gamma$, and regard it as a Riemann surface. Then the field of meromorphic functions on $\mathcal{M}_\Gamma$ is an algebraic function field.

There is a one to one correspondence

algebraic function fields

$\iff$ compact Riemann surfaces

What does it mean?

The surface is determined by the set of functions on it.

Question **How to generalize this?**

Answer **The solution space to the Helmholtz equation.**
More precisely, the behavior at infinity of solutions at infinity.
This leads us to the S-matrix.

Singular points

We need one more classification of action.

An element $\gamma \in \Gamma$ is said to be elliptic $\Longleftrightarrow \exists 1$ fixed point $\in \mathbb{C}_+$

$\iff |\mathrm{tr}\,\gamma| < 2.$

For the fixed point p, the isotropy group is defined by

$$\mathcal{I}(p) = \{\gamma \in \Gamma\,;\, \gamma \cdot p = p\}\,.$$

What we want to say next is

> **Around elliptic fixed points, $\mathcal{M}_\Gamma$ looks like a cone, whose vertex is therefore a singular point.**

One must be careful about the meaning of *singular*.

- **$\mathcal{M}_\Gamma$ is a Riemann surface without singular point as a 1-dim. complex manifold.**

- $\mathcal{M}_\Gamma$ can be regarded as a Riemannian manifold equipped with the hyperbolic metric.
- However, at the elliptic fixed points, this metric becomes singular.

In fact, around ellptic fixed points, $\mathcal{M}_\Gamma$ has the following

Orbifold structure

- By a suitable choice of local coordinates around $p \in \mathcal{M}_{sing}$ = the set of all elliptic singular points , the isotropy group $\mathcal{I}(p)$ turns out to be a finite rotation group.
- Then one can take a neighborhood of $p \in \mathcal{M}_{sing}$, which is like a sector with vertex at p.

- **Hence $\mathcal{M}$ admits a local covering space around p, which is isometric to the hyperbolic space.**

Around elliptic singular points, by a suitable change of variables, (the resulting local coordinates are no longer analytic at p),

$$ds^2 = (dr)^2 + \frac{1}{m^2}(\sin hr)^2(d\theta)^2, \quad m = \sharp\mathcal{I}(p).$$

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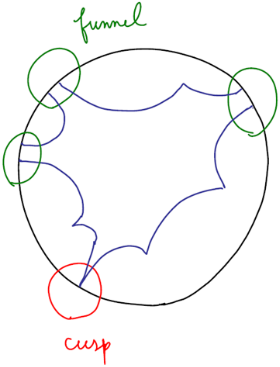
Most general example



§2 Surfaces with conical singularties

We introduce a surface which generalizes the fundamental domain of geometrically finite Fuchsian group.

§2.1 Assumptions

We consider a 2-dim. connected C^∞ manifold $\mathcal{M}$ which is written as a union of open sets,

$$\mathcal{M} = \mathcal{K} \cup \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_N$$

satisfying the following 4 assumptions:

(A-1) There exists $1 \leq \mu \leq N$ such that for $1 \leq i \leq \mu$, $\mathcal{M}_i$ is isometric to $S^1 \times (1, \infty)$ equipped with the metric

$$ds^2 = \frac{(dx)^2 + (dy)^2}{y^2}.$$

(So, $\mathcal{M}_1, \cdots, \mathcal{M}_\mu$ have cusps at infinity.)

(A-2) For $\mu + 1 \leq i \leq N$, $\mathcal{M}_i$ is diffeomorphic to $S^1 \times (0,1)$, and the metric on it has the following form

$$ds^2 = \frac{(dy)^2 + (dx)^2 + A(x,y,dx,dy)}{y^2},$$

$$A = a(x,y)(dx)^2 + 2b(x,y)dxdy + c(x,y)(dy)^2,$$

where a, b, c satisfy

$$|\partial_x^\alpha (y\partial_y)^n d(x,y)| \leq C_{\alpha n}(1 + |\log y|)^{-m(\alpha,n)-1-\epsilon},$$

for some $\epsilon > 0$,

$$m(\alpha, n) = \min(|\alpha| + n, 1).$$

(Therefore, for $\mu + 1 \leq i \leq N$, $\mathcal{M}_i$ have the (perturbed) regular infinity at $y = 0$.)

(A-3) $\overline{\mathcal{K}}$ is compact.

(A-4) There exists a finite subset $\mathcal{M}_{sing} \subset \mathcal{K}$ such that $\mathcal{M}$ has a C^∞ Riemannian metric g on $\mathcal{M} \setminus \mathcal{M}_{sing}$. To each $p \in \mathcal{M}_{sing}$, there exists an open set $\widetilde{U}_p \subset \mathrm{R}^2$ such that $0 \in \widetilde{U}_p$ and $\widetilde{U}_p$ has the metric $\widetilde{g}_p$ with the following conical structure : By the geodesic polar coordinates

$$\widetilde{g}_p = (dr)^2 + C_p r^2 (1 + h_p(r,\theta))(d\theta)^2,$$

$$0 < r < \epsilon, \quad 0 \leq \theta < 2\pi,$$

$$h_p(r,\theta) \to 0 \quad (r \to 0),$$

$$C_p > 0, \quad C_p \neq 1.$$

The non-compact parts $\mathcal{M}_j$ are called "ends".

§2.2 *Basic spectral properties*

Let Δ_g be the Laplace-Beltrami operator on $\mathcal{M}$, and put

$$H = -\Delta_g - \frac{1}{4}.$$

Its self-adjoint realization is defined through the quadratic form.

Lemma

$\sigma_{ess}(H) = [0, \infty)$.

§2.3 *Helmholtz equation and the "geometric" S-matrix*

As in the case of R^n, to solve the Helmholtz equation on $\mathcal{M}$, we introduce the Besov space $\mathcal{B}^*$. You can

imagine it easily, if you see the case of
$\mathcal{M} = S^1 \times (0, \infty)$:

$$u \in \mathcal{B}^* \Longleftrightarrow \sup_{R>e} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|u(\cdot, y)\|^2_{L^2(S^1)} \frac{dy}{y^2} < \infty.$$

We write

$$u \simeq v \quad (u \text{ and } v \text{ are similar at infinity})$$

if $u - v$ satisfies on each end

$$\lim_{R\to\infty} \frac{1}{\log R} \int_{\frac{1}{R}<y<R} \|u(\cdot, y) - v(\cdot, y)\|^2_{L^2(S^1)} \frac{dy}{y^2} = 0.$$

Then the space of physical solutions of the Helmholtz equation is

$$\mathcal{H}(k) = \left\{ u \in \mathcal{B}^* ; \, (H - k^2)u = 0 \right\}, \quad k^2 \in (0, \infty) \backslash \sigma_p(H).$$

The L^2-space at infinity is

$$\begin{cases} \mathbf{C} & \text{for the cusp,} \\ L^2(S^1) & \text{for the regular infinity.} \end{cases}$$

Therefore, the space of scattering data at infinity is

$$\mathrm{h}_\infty = \sum_{j=1}^{\mu} \mathbf{C} \oplus \sum_{j=\mu+1}^{N} \mathrm{L}^2(\mathrm{S}^1).$$

Here, let us recall that the counter part of the Euclidean spherical wave

$$\frac{e^{ikr}}{r^{(n-1)/2}}, \quad \text{in} \quad \mathbf{R}^n$$

is, in hyperbolic space,

$$y^{(n-1)/2 \mp ik}, \quad \text{in} \quad \mathbf{H}^n.$$

Letting $\{\chi_j\}$ be the partition of unity on $\mathcal{M}$, where χ_j localizes on $\mathcal{M}_j$, we see that any $u \in \mathcal{H}(k^2)$ admits the following asymptotic expansion :

$$\begin{aligned} u \simeq \omega_-(k) \sum_{j=1}^{\mu} \chi_j y^{1/2+ik} \psi_j^{(-)} \\ + \omega_-^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j y^{1/2-ik} \psi_j^{(-)} \\ - \omega_+(k) \sum_{j=1}^{\mu} \chi_j y^{1/2+ik} \psi_j^{(+)} \\ - \omega_+^{(c)}(k) \sum_{j=\mu+1}^{N} \chi_j y^{1/2+ik} \psi_j^{(+)}, \end{aligned}$$

$$\psi^{(\pm)} = (\psi_1^{(\pm)}, \cdots, \psi_N^{(\pm)}) \in \mathrm{h}_\infty.$$

The "geometric" S-matrix is then defined by

$$S(k) : \mathrm{h}_\infty \ni \psi^{(-)} \to \psi^{(+)} \in \mathrm{h}_\infty,$$

which is unitary on h_∞. It is a $N \times N$ matrix

$$S(k) = \Big(S_{ij}(k)\Big)$$

whose ij entry is a bounded operator

$$S_{ij}(k) : \mathrm{h_i} \to \mathrm{h_j},$$

where

$$\mathrm{h}_i = \begin{cases} \mathrm{C} & \text{if } \mathcal{M}_i \text{ is cusp}, \\ L^2(S^1) & \text{if } \mathcal{M}_i \text{ is regular infinity}. \end{cases}$$

§2.4 *Inverse scattering from regular ends*

Suppose we are given 2 such manifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$, and the associated geometric S-matrices $S^{(1)}(k), S^{(2)}(k)$. Suppose for some n_1, n_2, the ends $\mathcal{M}^{(1)}_{n_1}$ and $\mathcal{M}^{(2)}_{n_2}$ have regular infinity, and they are isometric. We assume, furthermore

$$S^{(1)}_{n_1 n_1}(k) = S^{(2)}_{n_2 n_2}(k), \quad \forall k^2 > 0.$$

Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric, and the conical structure around singular points coincide.

Therefore, the knowledge of the "geometric" S-matrix at regular infinity determines the manifolds.

§3 Inverse scattering from cusp

We are interested in the inverse scattering from cusp, since in some cases (e.g. Fuchsian groups of 1st kind), the infinity consists only of cusp.

§3.1 *Generalized S-matrix*

The cusp gives only 1-dimensional contribution to the continuous spectrum of H. Therefore, the component of the S-matrix corresponding to cusp, being merely a complex number, is not sufficient to determine the whole manifold.

The remedy consists in enlarging the solution space of the Helmholtz equation.

On th end $\mathcal{M}_1$, having a cusp, the Helmholtz equation takes the form

$$-y^2(\partial_y^2 + \partial_x^2)u - \frac{1}{4}u = k^2 u.$$

Expand u into a Fourier series

$$u(x,y) = \sum_{n\in \mathbf{Z}} e^{2\pi i n x} u_n(y),$$

Then

$$y^2(-\partial_y^2 + (2ny)^2)u_n - \frac{1}{4}u_n = k^2 u_n,$$

$$u_n(y) = \begin{cases} \tilde{a}_n y^{\frac{1}{2}} I_{-ik}(2\pi|n|y) + \tilde{b}_n y^{\frac{1}{2}} K_{ik}(2\pi|n|y), \ n \neq 0, \\ a_0 y^{\frac{1}{2}-ik} + b_0 y^{\frac{1}{2}+ik}, \ n = 0. \end{cases}$$

Here I_ν and K_ν are modified Bessel functions behaving like

$$I_\nu(z) \sim \frac{1}{\sqrt{2\pi z}} e^z, \quad z \to \infty,$$

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad z \to \infty,$$

One can construct a solution of the Helmholtz equation, which is exponentially growing as above in the cusp, and behaves like $y^{1/2\mp ik}$ in the regular infinity. Moreover at the cusp, it behaves like

$$u(x,y) \sim a_0 y^{1/2-ik} + \sum_{n\neq 0} a_n e^{inx+|n|y} + b_0 y^{1/2+ik} + \sum_{n\neq 0} b_n e^{inx-|n|y}.$$

We call the map

$$\mathcal{S}_{11}(k) : \{a_n\}_{n\in \mathbf{Z}} \to \{b_n\}_{n\in \mathbf{Z}}$$

the ((11)-entry of) generalized S-matrix.

Remark 1 **This is now a standard idea, in the study of inverse problems, of employing exponentially growing solutions for the Helmholtz equation.**

Remark 2 **The generalized S-matrix is an infinte matrix, and the usual S-matrix is its 00-entry.**

Remark 3 **We have an associated Poisson type integral formula for the Helmholtz equation. However, at the cusp, identifying the sequence with the Fourier series, we are dealing with a class of analytic functional bigger than Sato's hyperfunction.**

§3.2 Main Theorem

Suppose we are given 2 such manifolds $\mathcal{M}^{(1)}, \mathcal{M}^{(2)}$, and the associated generalized S-matrices

$\mathcal{S}^{(1)}(k), \mathcal{S}^{(2)}(k)$. Suppose both of them have a cusp for the end $\mathcal{M}_1^{(1)}$, $\mathcal{M}_1^{(2)}$, and assume, furthermore

$$\mathcal{S}_{11}^{(1)}(k) = \mathcal{S}_{11}^{(2)}(k), \quad \forall k^2 \in (0,\infty)\setminus\big(\sigma_p(H^{(1)})\cup\sigma_p(H^{(2)})\big).$$

Then $\mathcal{M}^{(1)}$ and $\mathcal{M}^{(2)}$ are isometric, and the conical structure around singular points coincide.

In particular, the geometrically finite Fuchsuan groups, (e.g. Fuchsian groups of the 1st kind), are determined by their generalized S-matrices.

§3.3 *Forward problem*

To study the forward problem, one needs to investigate the resolvent. The crucial steps are

- **Limiting absorption principle**
  i.e. the existence of the boundary values of the resolvent

$$R(k^2 \pm i0) \in \mathrm{B}(\mathcal{B}; \mathcal{B}^*),$$

- **Spectral representation**
  i.e. the construction of the partial isometry

$$\mathcal{F}^{(\pm)} : L^2(\mathcal{M}) \to L^2((0,\infty); \mathrm{h}_\infty; dk),$$

whcih diagonalizes H

$$\left(\mathcal{F}^{(\pm)} H f\right)(k) = k^2 \left(\mathcal{F}^{(\pm)} f\right)(k).$$

- **Asymptotic expansion of the resolvent at infinity**

$$R(k^2 \pm i0) f \simeq C_\pm(k) y^{\frac{1}{2} \mp ik} (\mathcal{F}^{(\pm)} f)(k).$$

They are carried out by rather elementary tools of integration by parts and asymptotic expansion of Bessel functions.

§3.4 *Boundary control method*

In the inverse procedure, we use

Boundary control method

invented by

M. Belishev, An approach to multidimensional inverse problems for the wave equation, Dokl. Akad. Nauk SSSR, 297, (1987), 524-527 (Engl. transl. Soviet Math. Dokl. 36 (1988), 481-484.

M. Belishev and V. Kurylev, To the reconstruction of a Riemannian manifold via its spectral data (BC-method), Comm. in P. D. E. 17 (1992), 767-804.

A. Katchalov, Y. Kurylev and M. Lassas, Inverse Boundary Spectral Problems, Chapman and Hall/CRC, Monographs and Surveys in Pure and Applied Mathematics, 123 (2001).

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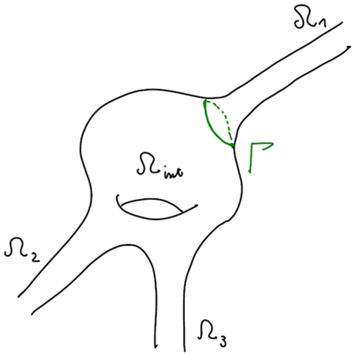
“Interior” boundary value problem



§3.5 *Works in progress*

We are trying to extend our results in higher dimensions. In 3-dimensions, you can let $SL(2, \mathrm{C})$ act on R^3_+ by using quartenions. By choosing discrete subgroups of $SL(2, \mathrm{C})$, one can construct interesting examples of 3-dim. hyperbolic orbifolds. For example, the one constructed by the Picard group

$$SL(2, \mathrm{Z} + i\mathrm{Z})\backslash \mathrm{H}^3$$

is a 3-dim. analogue of the modular surface

$$SL(2, \mathrm{Z})\backslash \mathrm{H}^2.$$

I feel very much honored to give a talk in this conference dedicated to Prof. Helgason.

Thank you for your attention and the warm hospitality!