Injectivity and Exact Inversion of Ultrasound Operators in the Spherical Geometry

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Geometric Analysis on Euclidean and Homogeneous Spaces
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Acknowledgements

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  - Venky Krishnan
  - Matthew Lewis
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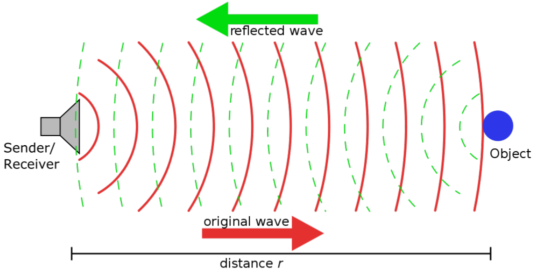
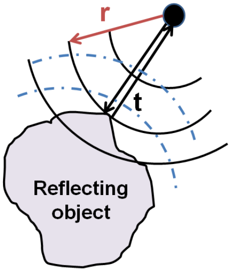
Ultrasound Reflection Tomography



- The pulse radiates isotropically
- Sound speed c is constant
- The medium is weakly reflecting
- The transducer is focused to receive signals only from a plane.

Mono-static setup

- The emitter coincides with the receiver.
- At any given moment of time t echoes are simultaneously produced along the sphere of radius $r = \frac{tc}{2}$.



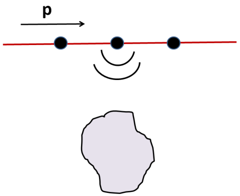
Circular Radon Transform

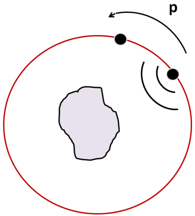
Definition

The circular Radon transform of f is defined as

$$Rf(p,r)=\int_{|x-p|=r} f(x)\,dl(x),$$

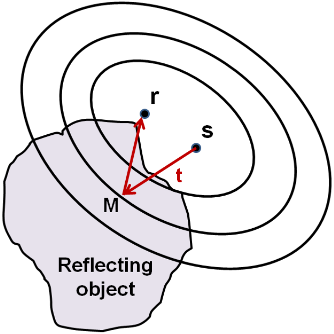
where $dl(x)$ is the arc length measure on the circle $|x-p|=r$.





Bi-static setup

- The emitter and the receiver are a fixed distance apart.
- At any given moment of time t echoes are simultaneously produced along confocal ellipses defined by $r_1 + r_2 = tc$.



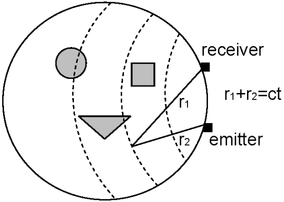
Elliptical Radon Transform

Definition

The elliptical Radon transform of a function $f(x)$ is defined as

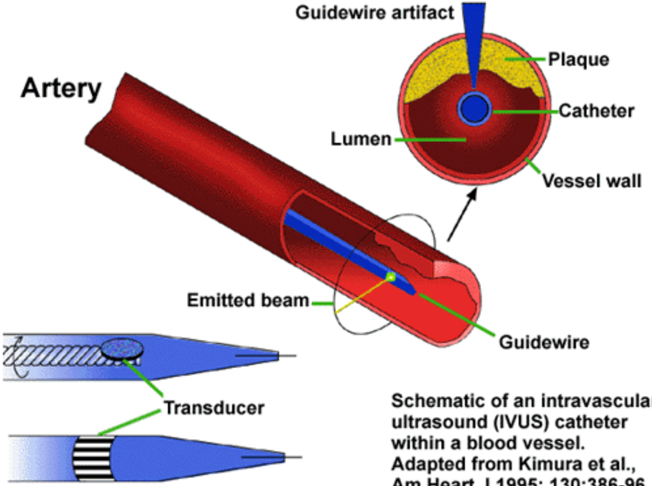
$$\tilde{\mathcal{R}}f(p_e, p_r, r) = \int_{|x-p_e|+|x-p_r|=r} f(x)d\sigma(x),$$

where $d\sigma(x)$ is the arc length measure on the ellipse $|x - p_e| + |x - p_r| = r$.



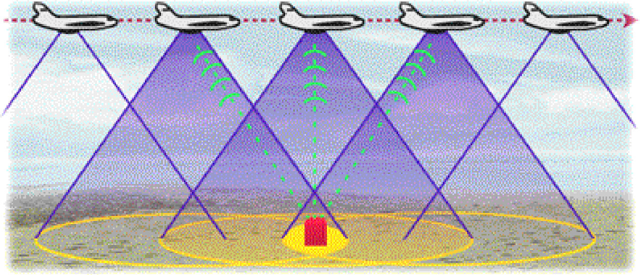
Rotating the transducer around the object

Intravascular Ultrasound (exterior support)



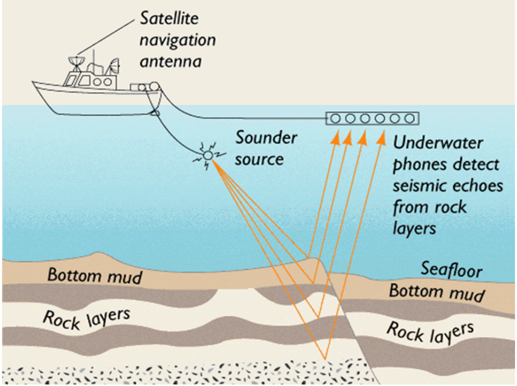
Schematic of an intravascular ultrasound (IVUS) catheter within a blood vessel. Adapted from Kimura et al., Am Heart J 1995; 130:386-96.

Radio Detection and Ranging (RADAR)



Synthetic Aperture Radar

Sound Navigation and Ranging (SONAR)



SONAR and Geophysical Exploration

Uniqueness of Reconstruction

For which values of p and r does the knowledge of $Rf(p,r)$ allow unique recovery of $f(x)$?
Similarly, for which values of p_e, p_r, and r does the knowledge of $\tilde{\mathcal{R}}f(p_e,p_r,r)$ allow unique recovery of $f(x)$?

Example

If $Rf(p,r)$ is known for p restricted to a line and all values of r, then f can not be uniquely recovered.

Definition

The transform Rf is called **injective** on a set $S \times T$ and functional class $\mathfrak{C}$ if for any $f \in \mathfrak{C}$ the equality $Rf(p,r) = 0$ for all $p \in S$ and all $r \in T$ implies $f \equiv 0$. In case of the elliptical transform we take $p_e,\ p_r \in S$, and the rest of the definition is the same.

The Spherical Transform with Full Radial Data

In the case of $T = \mathbb{R}$

- M. Agranovsky and E. T. Quinto (1996) gave a complete characterization of injectivity sets $S \subset \mathbb{R}^2$ when $\mathfrak{C} = C_c(\mathbb{R}^2)$.
- In dimensions higher than $n = 2$ or when the functions are not compactly supported a similar complete description of non-injectivity sets is not known. However, various necessary conditions for a set S to be a non-injectivity set for Rf have been provided by several groups (M. Agranovsky, C. Berenstein, and P. Kuchment (1996), D. Finch, Rakesh, and S. Patch (2004), G. A. and P. Kuchment (2005), ...).

Rf in Spherical Geometry with Radially Partial Data

If S is a sphere, then one can get unique reconstruction of f using data with limited radii.

M. Lavrentiev, M. Romanov, and M. Vasiliev (1970)

V. Volchkov (2003)

D. Finch, Rakesh, and S. Patch (2004):

M. Anastasio et al. (2005)

E. T. Quinto (2006), and M. Agranovsky, E.T. Quinto (1996)

P. Stefanov, and G. Uhlmann (2011)

Reconstruction formulas

Theorem (D. Finch, M. Haltmeier, and Rakesh (2007))

Let $D \subset \mathbb{R}^2$ be the disk of radius R centered at the origin, and let $f \in C^\infty(\mathbb{R}^2)$ with $\operatorname{supp} f \subset \bar{D}$. Then, for $x \in D$,

$$f(x) = \frac{1}{2\pi R} \triangle_x \int_0^{2\pi} \int_0^{2R} \rho\, g(\rho, \phi) \log \left| \rho^2 - |x - p|^2 \right| \, d\rho\, d\phi$$

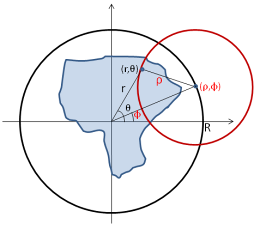
- Notice, that the knowledge of the Radon transform for all $\rho > 0$ is required.
- The formula does not hold if *supp f* is not a subset of $\bar{D}$.

Reconstruction formulas

Fourier expansion methods

$$g(\rho,\phi)=\sum_{n=-\infty}^{\infty} g_n(\rho)e^{-in\phi} \qquad g_n(\rho)=\frac{1}{2\pi}\int_0^{2\pi} g(\rho,\phi)e^{-in\phi}\,d\phi$$

$$f(r,\theta)=\sum_{n=-\infty}^{\infty} f_n(r)e^{-in\theta} \qquad f_n(r)=\frac{1}{2\pi}\int_0^{2\pi} f(r,\theta)e^{-in\theta}\,d\theta$$



Reconstruction formulas

Theorem (S. J. Norton (1979))

If $f(r,\theta)$ is supported inside the disc of radius R, then one can recover its Fourier coefficients $f_n(r)$ from Fourier coefficients $g_n(\rho)$ of its circular Radon transform $g(\rho,\phi) = Rf(\rho,\phi)$ as follows:

$$f_n(r) = \mathcal{H}_n \left\{ \frac{1}{J_n(Rz)} \mathcal{H}_0 \left\{ \frac{g_n(\rho)}{2\pi\rho} \right\}_z \right\}_r,$$

where $\mathcal{H}_n$ is the n^{th} order Hankel transform defined by

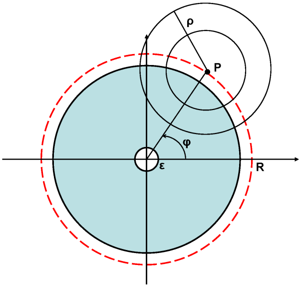
$$(\mathcal{H}_n h)(\sigma) = \int_0^\infty J_n(\sigma r)\, h(r)\, r\, dr.$$

Notice, the knowledge of the Radon transform for all $\rho > 0$ is required.

Reconstruction from Partial Data - Interior Problem

Theorem (G. A., R. Gouia, and M. Lewis (2010))

Let $f(r,\theta)$ be an unknown continuous function supported inside the annulus $A(\varepsilon, R) = \{(r,\theta) : r \in (\varepsilon, R), \theta \in [0, 2\pi]\}$, where $0 < \varepsilon < R$. If $Rf(\rho,\phi)$ is known for $\phi \in [0, 2\pi]$ and $\rho \in [0, R-\varepsilon]$, then $f(r,\theta)$ can be uniquely recovered in $A(\varepsilon, R)$.



Reconstruction Formulas - Exterior Problem

Define

$$F_n(u) := f_n(R-u) \tag{1}$$

$$K_n(\rho,u) := \frac{4\rho\,(R-u)\;T_{|n|}\left[\frac{(R-u)^2+R^2-\rho^2}{2R(R-u)}\right]}{\sqrt{(u+\rho)(2R+\rho-u)(2R-\rho-u)}} \tag{2}$$

$$G_n(t) := \frac{1}{\pi K_n(t,t)}\,\frac{d}{dt}\int_0^t \frac{g_n(\rho)}{\sqrt{t-\rho}}\,d\rho \tag{3}$$

$$L_n(t,u) := \frac{1}{\pi K_n(t,t)}\,\frac{\partial}{\partial t}\int_u^t \frac{K_n(\rho,u)}{\sqrt{\rho-u}\sqrt{t-\rho}}\,d\rho \tag{4}$$

Theorem (G. A., R. Gouia, and M. Lewis (2010))

An exact formula expressing the Fourier coefficients of the function f by those of Rf is given by

$$F_n(t) = G_n(t) + \int_0^t H_n(t,u)\, G_n(u)\, du, \tag{5}$$

where $H_n(t,u)$ is given by the series of iterated kernels

$$H_n(t,u) = \sum_{i=1}^{\infty} (-1)^i L_{n,i}(t,u), \tag{6}$$

defined by

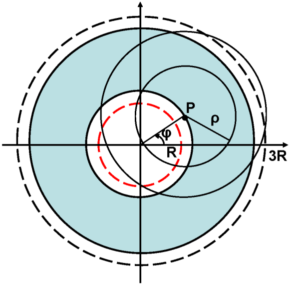
$$L_{n,1}(t,u) = L_n(t,u), \tag{7}$$

$$L_{n,i}(t,u) = \int_u^t L_{n,1}(t,x)\, L_{n,i-1}(x,u)\, dx, \quad \forall\, i \geq 2. \tag{8}$$

Reconstruction from Partial Data - Exterior Problem

Theorem (G. A., R. Gouia, and M. Lewis (2010))

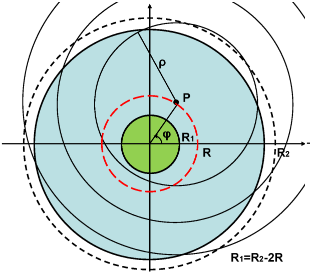
Let $f(r,\theta)$ be an unknown continuous function supported inside the annulus $A(R,\ 3R) = \{(r,\theta) : r \in (R,\ 3R),\ \theta \in [0, 2\pi]\}$. If $Rf(\rho,\phi)$ is known for $\phi \in [0, 2\pi]$ and $\rho \in [0, R_1]$, where $0 < R_1 < 2R$ then $f(r,\theta)$ can be uniquely recovered in $A(R,\ R_1)$.

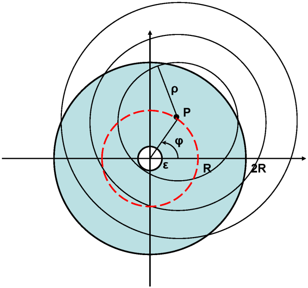


Support Both Inside and Outside

Theorem (G. A. and V. Krishnan (2011))

Let $f(r,\theta)$ be an unknown continuous function supported inside the disc $D(0,R_2)$, where $R_2 > 2R$. If $Rf(\rho,\phi)$ is known for $\phi \in [0,2\pi]$ and $\rho \in [R_2 - R, R_2 + R]$, then $f(r,\theta)$ can be uniquely recovered in $A(R_1,\ R_2)$, where $R_1 = R_2 - 2R$.

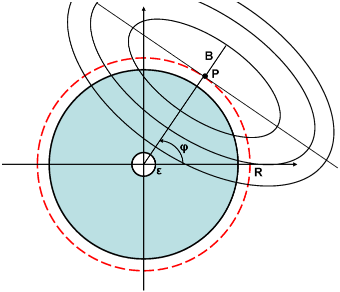




Reconstruction from Partial Data - Interior Problem

Theorem (G. A. and V. Krishnan (2011))

Let $f(r,\theta)$ be an unknown continuous function supported inside the annulus $A(\varepsilon, R) = \{(r,\theta) : r \in (\varepsilon, R), \theta \in [0, 2\pi]\}$, where $0 < \varepsilon < R$. If $\tilde{\mathcal{R}}f(B,\phi)$ is known for $\phi \in [0, 2\pi]$ and $B \in [0, R-\varepsilon]$, then $f(r,\theta)$ can be uniquely recovered in $A(\varepsilon, R)$.



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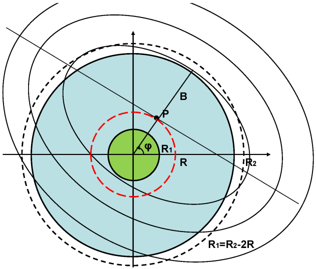

Thank you for your attention!