The bounded hypergeometric functions associated with root systems

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The bounded spherical functions

G/K = Riemannian symmetric space of the noncompact type

G = connected noncompact semisimple Lie group with finite center
K = maximal compact subgroup of G

Spherical functions = (normalized) K-invariant joint eigenfunctions of the commutative algebra of G-invariant differential operators on G/K

$\rightsquigarrow$ building blocks of the K-invariant harmonic analysis on G/K

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition of the Lia algebra of G
$\mathfrak{a} \subset \mathfrak{p}$ maximal abelian subspace (Cartan subspace)
Σ = (restricted) roots of $(\mathfrak{g}, \mathfrak{a})$
W = Weyl group of Σ

$\rightsquigarrow$ spherical functions are parametrized by $\mathfrak{a}_{\mathbb{C}}^*$ (modulo W)

Σ^+ = choice of positive roots in Σ
m_α = multiplicity of the root $\alpha \in \Sigma$
$\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$

Harish-Chandra's integral formula: $\varphi_\lambda(gK) = \int_K e^{(\lambda-\rho)(H(gk))}\, dk\,, \quad g \in G\,,$
where $H(x) \in \mathfrak{a}$ is the Iwasawa component of $x \in G = KAN$
Then: $\varphi_{w\lambda} = \varphi_\lambda$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $w \in W$.

Recall $\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$
$C(\rho)$ = convex hull in $\mathfrak{a}^*$ of $\{w\rho : w \in W\}$

Theorem (Helgason & Johnson, 1969)

The spherical function φ_λ (with $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$) is bounded if and only if $\lambda \in C(\rho) + i\mathfrak{a}^*$.

In this talk:

1. Extend the Helgason-Johnson's theorem to the hypergeometric functions associated with root systems (theory of Heckman and Opdam)
2. Applications to L^p-theory of the hypergeometric Fourier transform ($1 \leq p < 2$).

Heckman-Opdam's hypergeometric functions

- The symmetric space G/K is replaced by a triple $(\mathfrak{a}, \Sigma, m)$ where:
  $\mathfrak{a}$ = finite dim. Euclidean $\mathbb{R}$-vector space, inner product $\langle \cdot, \cdot \rangle$
  Σ = root system in $\mathfrak{a}^*$, with Weyl group W
  m = positive multiplicity function on Σ
  i.e. $m : \Sigma \to [0, +\infty[$, W-invariant: $m_{w\alpha} = m_\alpha$ for all $\alpha \in \Sigma$, $w \in W$

$(\mathfrak{a}, \Sigma, m)$ is *geometric* if associated with a Riemannian symmetric space of the noncompact type

- Commutative family $\mathbb{D}$ of differential operators associated with $(\mathfrak{a}, \Sigma, m)$:

For $x \in \mathfrak{a}$ the *Cherednik operator* T_x is the difference-reflection operator on $\mathfrak{a}$ (or $\mathfrak{a}_\mathbb{C}$) defined for $f \in C^\infty(\mathfrak{a})$ and $H \in \mathfrak{a}$ by

$$T_x f(H) = \partial_x f(H) + \sum_{\alpha \in \Sigma^+} m_\alpha \alpha(x) \frac{f(H) - f(r_\alpha H)}{1 - e^{-2\alpha(H)}} - \rho(x) f(H)$$

where r_α = reflection across $\ker \alpha$.

$\{x \in \mathfrak{a} \mapsto T_x\}$ commutative $\Rightarrow$ extends as algebra homomorphism $\{p \in S(\mathfrak{a}_\mathbb{C}) \mapsto T_p\}$

If $p \in S(\mathfrak{a}_\mathbb{C})^W$, then $D_p := T_p|_{C^\infty(\mathfrak{a})^W}$ is a differential operator on $\mathfrak{a}$ (or $\mathfrak{a}_\mathbb{C}$).

$\mathbb{D} = \mathbb{D}(\mathfrak{a}, \Sigma, m) := \{D_p : p \in S(\mathfrak{a}_\mathbb{C})^W\}$

- *Hypergeometric function* of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$:

unique W-invariant analytic function F_λ on $\mathfrak{a}$ which satisfies the system of diff eqs

$$\begin{aligned} D_p F_\lambda &= p(\lambda) F_\lambda\,, \qquad p \in S(\mathfrak{a}_{\mathbb{C}})^W\,, \\ F_\lambda(0) &= 1 \end{aligned}$$

Then: $F_{w\lambda} = F_\lambda$ for all $w \in W$.

Examples:

(1) $(\mathfrak{a}, \Sigma, m)$ geometric: $\mathfrak{a} \equiv \exp \mathfrak{a} \cdot o \subset G/K$
$\mathbb{D} \equiv$ radial components on $\mathfrak{a}^+$ of the G-invariant differential operators on G/K
$F_\lambda \equiv \varphi_\lambda$ Harish-Chandra's spherical function of spectral parameter $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

(2) rank-one (i.e. $\dim_{\mathbb{R}} \mathfrak{a} = 1$): Jacobi function of 2nd kind

$$F_\lambda(x) = {}_2F_1\left(\frac{m_\alpha/2 + m_{2\alpha} + \lambda}{2}, \frac{m_\alpha/2 + m_{2\alpha} - \lambda}{2}; \frac{m_\alpha/2 + m_{2\alpha} + 1}{2}; -\sinh^2 x\right)$$

- *Nonsymmetric hypergeometric function* of spectral param $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ (Opdam, 1995):

unique analytic function G_λ on $\mathfrak{a}$ which satisfies the system of diff-difference equations

$$\begin{aligned} T_x G_\lambda &= \lambda(x) G_\lambda\,, \qquad x \in \mathfrak{a}\,, \\ G_\lambda(0) &= 1 \end{aligned}$$

- Relation: $F_\lambda(x) = \frac{1}{|W|} \sum_{w \in W} G_\lambda(wx)\,.$
- Basic estimate (Schapira, 2008):
  (1) F_λ real and positive if $\lambda \in \mathfrak{a}^*$
  (2) $|F_\lambda| \leq F_{\mathrm{Re}\,\lambda}$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$

The Harish-Chandra series Φ_λ

Solution of the hypergeometric system of differential equations:

$$D_p\Phi = p(\lambda)\Phi\,, \qquad p \in S(\mathfrak{a}_\mathbb{C})^W$$

of the form:

$$\Phi_\lambda(x) = e^{(\lambda-\rho)(x)} \sum_{\mu\in\Lambda} \Gamma_\mu(\lambda) e^{-\mu(x)}\,, \qquad x \in \mathfrak{a}^+$$

where:

$\Lambda = \left\{ \sum_{j=1} n_j\alpha_j : n_j \in \mathbb{N}_0 \right\}$ and $\{\alpha_1, \dots, \alpha_l\}$ basis simple roots associated with Σ^+,

$\Gamma_\mu(\lambda) =$ rational functions determined by the recursion relations

$$\Gamma_0(\lambda) = 1$$

$$\langle \mu, \mu - 2\lambda \rangle \Gamma_\mu(\lambda) = 2 \sum_{\alpha\in\Sigma^+} m_\alpha \sum_{\substack{k\in\mathbb{N} \\ \mu-2k\alpha\in\Lambda}} \Gamma_{\mu-2k\alpha}(\lambda) \langle \mu + \rho - 2k\alpha - \lambda, \alpha \rangle$$

Then $\Gamma_\mu(\lambda) = 0$ for $\mu \in \Lambda \setminus 2\Lambda$. Many singularities are in fact removable.

Notation: $\mathcal{H}_{\alpha,r} = \{\lambda \in \mathfrak{a}_\mathbb{C}^* : \lambda_\alpha = r\}$ where $\lambda_\alpha = \frac{\langle\lambda,\alpha\rangle}{\langle\alpha,\alpha\rangle}$

$\Sigma_0^+ = \{\alpha \in \Sigma^+ : \alpha/2 \notin \Sigma\}$

Theorem (Opdam 1989, Heckman 1994)

1. $\Gamma_\mu(\lambda)$ *has at most simple poles along* $\mathcal{H}_{\alpha,n}$ *with* $\alpha \in \Sigma_0^+$, $n \in \mathbb{N}$ *and* $2n\alpha \le \mu$.
2. *There is a tubular nbd* U^+ *of* $\mathfrak{a}^+$ *in* $\mathfrak{a}_\mathbb{C}$ *so that* $\Phi_\lambda(x)$ *is a meromorphic function of* $(\lambda, x) \in \mathfrak{a}_\mathbb{C}^* \times U^+$ *with at most simple poles along* $\mathcal{H}_{\alpha,n}$ *with* $\alpha \in \Sigma_0^+$ *and* $n \in \mathbb{N}$.

Recall: For $\alpha \in \Sigma$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, set $\lambda_\alpha = \frac{\langle \lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle}$

Harish-Chandra's c-function is the meromorphic function on $\mathfrak{a}_{\mathbb{C}}^*$ defined by

$$c(\lambda) = c_{HC} \prod_{\alpha \in \Sigma_0^+} \frac{2^{-\lambda_\alpha}\, \Gamma(\lambda_\alpha)}{\Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{1}{2}\right) \Gamma\left(\frac{\lambda_\alpha}{2} + \frac{m_\alpha}{4} + \frac{m_{2\alpha}}{2}\right)},$$

where Γ is the gamma function and c_{HC} is a constant so that $c(\rho) = 1$.

Definition: $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is *generic* if $\lambda_\alpha \notin \mathbb{Z}$ for all $\alpha \in \Sigma_0$.

Theorem (Heckman & Opdam, 1989; Opdam, 1995)

Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ be generic. Then:

1. $\{\Phi_{w\lambda}(x) : w \in W\}$ *is a basis of the C^∞ solution space of the hypergeometric system of spectral parameter λ on $\mathfrak{a}^+$.*
2. *For $x \in \mathfrak{a}^+$*

$$F_\lambda(x) = \sum_{w \in W} c(w\lambda) \Phi_{w\lambda}(x) .$$

Asymptotic expansion of F_λ

- If $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ is generic and $x \in \mathfrak{a}^+$, then

$$F_\lambda(x) = \sum_{w \in W} c(w\lambda)\Phi_{w\lambda}(x)$$

$$\Phi_\lambda(x) = e^{(\lambda-\rho)(x)} \sum_{\mu \in 2\Lambda} \Gamma_\mu(\lambda) e^{-\mu(x)} .$$

Hence

$$F_\lambda(x) = \sum_{\mu \in 2\Lambda} \sum_{w \in W} c(w\lambda)\Gamma_\mu(w\lambda) e^{(w\lambda-\rho-\mu)(x)} .$$

- If $\lambda = \lambda_0 \in \mathfrak{a}_{\mathbb{C}}^*$ is arbitrary (and WLOG $\langle \mathrm{Re}\,\lambda_0, \alpha\rangle \geq 0$ for all $\alpha \in \Sigma_0^+$),

define:

$$n_\alpha = (\lambda_0)_\alpha, \qquad \Sigma^0_{\lambda_0} = \{\alpha \in \Sigma_0^+ : n_\alpha = 0\}, \qquad \Sigma^+_{\lambda_0} = \{\alpha \in \Sigma_0^+ : n_\alpha \in \mathbb{N}\},$$

$$\pi_0(\lambda) = \prod_{\alpha \in \Sigma^0_{\lambda_0}} \langle \lambda, \alpha\rangle$$

$$p_{w,\pm}(\lambda) = \prod_{\alpha \in \Sigma^+_{\lambda_0} \cap w(\pm\Sigma_0^+)} (\langle \lambda, \alpha\rangle - n_\alpha \langle \alpha, \alpha\rangle)$$

} products of linear factors, all vanishing at λ_0

$$p(\lambda) = \pi_0(\lambda) p_{w,+}(\lambda) p_{w,-}(\lambda)$$ independent of w!

$$\pi(\lambda) = \pi_0(\lambda) \prod_{\alpha \in \Sigma^+_{\lambda_0}} \langle \lambda, \alpha\rangle$$ highest order term of $p(\lambda)$

$p(\lambda) = \pi_0(\lambda) p_{w,+}(\lambda) p_{w,-}(\lambda)$, product of linear factors, all vanishing at λ_0
$\pi(\lambda) =$ highest order term of $p(\lambda)$

Lemma

1. *There is a nbd U of λ_0 so that, for all $w \in W$ and $\mu \in 2\Lambda \setminus \{0\}$, the functions $\pi_0(\lambda) p_{w,-}(\lambda) c(w\lambda)$ and $p_{w,+}(\lambda) \Gamma_\mu(w\lambda)$ are holomorphic in U.*
2. *For all $x \in \mathfrak{a}$, we have*
$$c_0 F_{\lambda_0}(x) = \partial(\pi)\Big(p(\lambda) F_\lambda(x)\Big)\Big|_{\lambda=\lambda_0}$$
where $c_0 = \partial(\pi)(p) = \partial(\pi)(\pi) > 0$.
3. *Let $x_0 \in \mathfrak{a}^+$ be fixed. Then*
$$c_0 F_{\lambda_0}(x) = \sum_{\mu \in 2\Lambda} \sum_{w \in W} \partial(\pi)\Big(p(\lambda) c(w\lambda) \Gamma_\mu(w\lambda) e^{(w\lambda - \rho - \mu)(x)}\Big)\Big|_{\lambda=\lambda_0}$$
where the series on the right-hand side converges uniformly in $x \in x_0 + \overline{\mathfrak{a}^+}$.

Remark/example: For $\lambda_0 = 0$ we have

$$\Sigma^0_{\lambda_0} = \Sigma^+_0\,, \qquad \Sigma^+_{\lambda_0} = \emptyset\,, \qquad p(\lambda) = \pi(\lambda) = \prod\nolimits_{\alpha \in \Sigma^+_0} \langle \lambda, \alpha \rangle\,.$$

Then

$$c_0 F_0(\lambda) = \partial(\pi)\big(\pi(\lambda) F_\lambda\big)|_{\lambda=0} \quad \text{where} \quad c_0 = \partial(\pi)(\pi) > 0.$$

(Harish-Chandra, Anker, Schapira)

$W_{\lambda_0} := \{w \in W : w\lambda_0 = \lambda_0\}$.
$b_0(\lambda) := \pi_0(\lambda)c(\lambda) \quad \rightsquigarrow b_0(w\lambda_0) = b_0(\lambda_0) \neq 0$ for all $w \in W_{\lambda_0}$
$\rho_0 := \sum_{\alpha \in \Sigma^0_{\lambda_0}} \alpha \quad \rightsquigarrow \pi_0(\rho_0) > 0$

Theorem

Let $x_0 \in \mathfrak{a}^+$ be fixed. Then for $x \in x_0 + \overline{\mathfrak{a}^+}$ we have

$$c_0 F_{\lambda_0}(x) = \Big(\frac{c_0}{\pi_0(\rho_0)} b_0(\lambda_0)\pi_0(x) + f_{\lambda_0}(x)\Big) e^{(\lambda_0-\rho)(x)} + \sum_{w \in W \setminus W_{\lambda_0}} \Big(b_w(\lambda_0)\pi_{w,\lambda_0}(x) + f_{w,\lambda_0}(x)\Big) e^{(w\lambda_0-\rho)(x)} + \sum_{\mu \in 2\Lambda\setminus\{0\}} \sum_{w \in W} f_{w,\mu,\lambda_0}(x) e^{(w\lambda_0-\rho-\mu)(x)}$$

where:
$c_0 = \partial(p)(\pi) = \partial(\pi)(\pi) > 0$,
the constants $b_w(\lambda_0)$ and the polynomials $\pi_{w,\lambda_0}(x)$ are explicit,
$f_{\lambda_0}(x)$ is a polynomial function of x of degree $< \deg \pi_0(x)$,
$f_{w,\lambda_0}(x)$ is a polynomial function of x of degree $< \deg \pi_{w,\lambda_0}(x) = \deg \pi_0(x)$.
The series converges uniformly in $x \in x_0 + \overline{\mathfrak{a}^+}$.

The bounded hypergeometric functions

Theorem

The hypergeometric function F_λ is bounded if and only if $\lambda \in C(\rho) + i\mathfrak{a}^*$. Moreover, $|F_\lambda(x)| \leq 1$ for all $\lambda \in C(\rho) + i\mathfrak{a}^*$ and $x \in \mathfrak{a}$.

Proof (sketch).

$\Leftarrow$: (Argument due to E. M. Stein)
Apply the maximum modulus principle to $\lambda \mapsto F_\lambda(x)$ with $x \in \mathfrak{a}$ fixed.
Since $|F_\lambda| \leq F_{\mathrm{Re}\,\lambda}$, the max of this function in $C(\rho) + i\mathfrak{a}^*$ is attained at $w\rho$, $w \in W$.
To compute $F_{w\rho}(x) = F_\rho(x)$:
$G_{-\rho} \equiv 1$ (from differential-difference equations)
$w_0 =$ longest element in W. Then for all $x \in \mathfrak{a}$:

$$F_\rho(x) = F_{w_0\rho}(x) = F_{-\rho}(x) = |W|^{-1} \sum_{w \in W} G_{-\rho}(wx) = 1\,.$$

$\Rightarrow$: (use asymptotic expansion of F_λ)
If $\mathrm{Re}\,\lambda_0 \in \overline{(\mathfrak{a}^*)^+} \setminus C(\rho)$, then there is $x_1 \in \mathfrak{a}^+$ so that $(\mathrm{Re}\,\lambda_0 - \rho)(x_1) > 0$.
If F_{λ_0} bounded, then $\lim_{t \to +\infty} F_{\lambda_0}(tx_1) e^{-t(\mathrm{Re}\,\lambda_0 - \rho)(x_1)} t^{-d} = 0$.
Here $d := \deg \pi_0$.

$\lim_{t\to+\infty} F_{\lambda_0}(tx_1)e^{-t(\mathrm{Re}\,\lambda_0-\rho)(x_1)}t^{-d} = 0$ and $\pi_0(x_1) \neq 0$ as $x_1 \in \mathfrak{a}^+$.
Asymptotic expansion gives:

$$\Big| \frac{F_{\lambda_0}(tx_1)e^{-t(\mathrm{Re}\,\lambda_0-\rho)(x_1)}}{t^d\pi_0(x_1)} - \Big(\frac{b_0(\lambda_0)}{\pi_0(\rho_0)} e^{it\,\mathrm{Im}\,\lambda_0(x_1)} + \sum_{w\in W_{\mathrm{Re}\,\lambda_0}\setminus W_{\lambda_0}} \frac{b_w(\lambda_0)\pi_{w,\lambda_0}(x_1)}{c_0\pi_0(x_1)} e^{itw\,\mathrm{Im}\,\lambda_0(x_1)} \Big) \Big| = o(t) \qquad \text{as } t\to+\infty\,.$$

It follows that

$$\lim_{t\to+\infty} \Big(\frac{b_0(\lambda_0)}{\pi_0(\rho_0)} e^{it\,\mathrm{Im}\,\lambda_0(x_1)} + \sum_{w\in W_{\mathrm{Re}\,\lambda_0}\setminus W_{\lambda_0}} \frac{b_w(\lambda_0)\pi_{w,\lambda_0}(x_1)}{c_0\pi_0(x_1)} e^{itw\,\mathrm{Im}\,\lambda_0(x_1)} \Big) = 0\,.$$

Since $x_1 \in \mathfrak{a}^+$ we have $w\,\mathrm{Im}\,\lambda_0(x_1) \neq \mathrm{Im}\,\lambda_0(x_1)$ for all $w \in W_{\mathrm{Re}\,\lambda_0} \setminus W_{\lambda_0} \subset W \setminus W_{\mathrm{Im}\,\lambda_0}$.
The limit 0 is possible only if $\frac{b_0(\lambda_0)}{\pi_0(\rho_0)} = 0$. Contradiction. □

Applications: L^p-harmonic analysis

Hypergeometric Fourier transform of a (suff regular) W-invariant $f : \mathfrak{a} \to \mathbb{C}$:

$$\widehat{f}(\lambda) := \int_{\mathfrak{a}} f(x) F_\lambda(x) d\mu(x)\,, \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*\,,$$

where

$$d\mu(x) = \prod_{\alpha \in \Sigma^+} \left| e^{\alpha(x)} - e^{-\alpha(x)} \right|^{m_\alpha} dx.$$

Plancherel measure (Opdam, 1995): $d\nu(\lambda) = |c(\lambda)|^{-2}\, d\lambda\,.$

For $1 \leq p < 2$, set:

$\epsilon_p = \frac{2}{p} - 1$

$C(\epsilon_p \rho)$ = convex hull in $\mathfrak{a}^*$ of the set $\{\epsilon_p w\rho : w \in W\}$

$\mathfrak{a}_{\epsilon_p}^* = C(\epsilon_p\rho) + i\mathfrak{a}^*$

Corollary

Let $f \in L^1(\mathfrak{a}, d\mu)^W$. Then:

1. $\widehat{f}(\lambda)$ *is well def and continuous on* $\mathfrak{a}_{\epsilon_1}^* = C(\rho) + i\mathfrak{a}^*$, *holomorphic in its interior.*
2. $|\widehat{f}(\lambda)| \leq \|f\|_1$ *for* $\lambda \in \mathfrak{a}_{\epsilon_1}^*$.
3. (Riemann-Lebesgue) *We have* $\lim_{\lambda \in \mathfrak{a}_{\epsilon_1}^*, |\mathrm{Im}\,\lambda| \to \infty} |\widehat{f}(\lambda)| = 0\,.$

Corollary

Let $f \in L^p(\mathfrak{a}, d\mu)^W$ *with* $1 < p < 2$. *Then:*

1. $\widehat{f}(\lambda)$ *is well def and holomorphic in the interior of on* $\mathfrak{a}^*_{\epsilon_p} = C(\epsilon_p \rho) + i\mathfrak{a}^*$.
2. (Hausdorff-Young) *Let* $1/p + 1/q = 1$. *Then* $\exists C_p \geq 0$ *so that*
$$\|\widehat{f}(\lambda)\|_q = \Big(\int_{i\mathfrak{a}^*} |\widehat{f}(\lambda)|^q \, d\nu(\lambda) \Big)^{1/q} \leq C_p \|f\|_p \,.$$
3. (Riemann-Lebesgue) *We have* $\lim_{\lambda \in \mathfrak{a}^*, |\lambda| \to \infty} |\widehat{f}(i\lambda)| = 0$.

Rem: Hausdorff-Young is an application of Riesz-Thorin interpolation thm to $f \mapsto \widehat{f}$. This operator is of type $(2,2)$ by Plancherel (Opdam, 95) and of type $(1, \infty)$ by previous corollary.

Lemma (Flensted-Jensen & Koornwinder, 1973)

For $f \in L^p(\mathfrak{a}, d\mu)^W$, $1 \leq p < 2$, *and* $g \in C_c^\infty(\mathfrak{a})^W$*:* $\int_{i\mathfrak{a}^*} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)} d\nu(\lambda) = \int_{\mathfrak{a}} f(x)\overline{g(x)} d\mu(x)$

Rem: Consequence of Paley-Wiener and Plancherel (Opdam, 95), $\|\widehat{f}\|_\infty \leq \|f\|_1$ and Hausdorff-Young.

Corollary

1. *The hypergeometric Fourier transform is injective on* $L^p(\mathfrak{a}, d\mu)^W$.
2. *If* $f \in L^p(\mathfrak{a}, d\mu)^W$ *and* $\widehat{f} \in L^1(i\mathfrak{a}^*, d\nu)^W$, *then* $f(x) = \int_{i\mathfrak{a}^*} \widehat{f}(\lambda) F_{-\lambda}(x) d\nu(\lambda)$ *a.e.* x