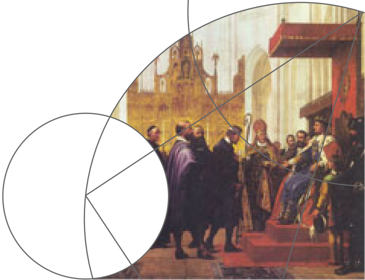
Department of Mathematical Sciences

Radon transformation on reductive symmetric spaces

support theorems

Job Kuit



Universiteit Utrecht

Horospherical transform on a Riemannnian symmetric space

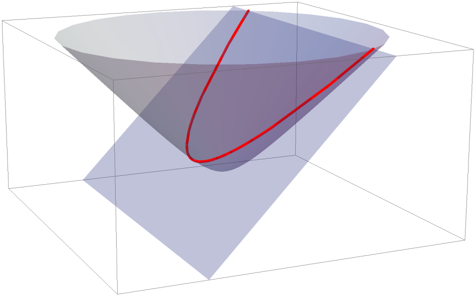
Riemannian symmetric space: $X = G/K$,
where G semisimple, K maximal compact subgroup

Horosphere: $gN \cdot K \subset X$,
where N unipotent radical of a minimal parabolic subgroup $P = MAN$

$\Xi = G/MN$ parameterizes the set of horospheres.

Example: Upper sheet of the hyperboloid

$G = SL(2, \mathbb{R})$,
$K = SO(2)$,
$$P = \left\{ \begin{pmatrix} a & b \\ 0 & \frac{1}{a} \end{pmatrix} : a, b \in \mathbb{R}, a \neq 0 \right\}$$
$$N = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{R} \right\}$$



$\langle \eta, x \rangle = p$, with $\eta \neq 0$ in the light cone

Double fibration:

$$\begin{array}{ccccc} & & Z = G/M & & \\ & \swarrow^{\Pi_X} & & \searrow^{\Pi_\Xi} & \\ G/K = X & & & & \Xi = G/MN \end{array}$$

Double fibration:

$$Z = G/M$$

$$G/K = X \xleftarrow{\ \Pi_X\ } Z \xrightarrow{\ \Pi_\Xi\ } \Xi = G/MN$$

Radon transforms

$$\mathcal{R}\phi : \Xi \ni g \cdot MN \mapsto \int_N \phi(gn \cdot K)\, dn \qquad \Big(\phi \in C_c^\infty(X)\Big)$$

$$\mathcal{R}^*\psi : X \ni g \cdot K \mapsto \int_{K/M} \psi(gk \cdot MN)\, dk \qquad \Big(\psi \in C^\infty(\Xi)\Big)$$

Support theorem

Let $\phi \in C_0^\infty(X)$, B ball in X

Support Theorem

$$\text{supp}(\phi) \subseteq B \quad \Longrightarrow \quad \mathcal{R}\phi(\xi) = 0 \text{ for all } \xi \cap B = \emptyset$$

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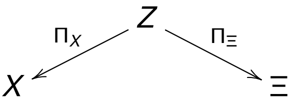
Support Theorem (Helgason, 1973)

$\operatorname{supp}(\phi) \subseteq B \iff \mathcal{R}\phi(\xi) = 0$ for all $\xi \cap B = \emptyset$

Horospherical transform for a reductive symmetric space



X: reductive symmetric space

Ξ: set of horospheres

Reductive symmetric spaces

G reductive Lie group of the Harish-Chandra class

σ involution on G

H open subgroup of G^σ satisfying some technical conditions

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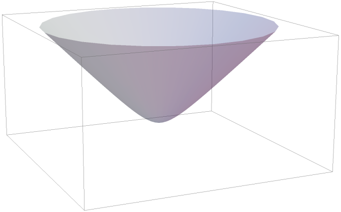
$X = G/H$ reductive symmetric space

Examples

$G = SL(2, \mathbb{R})$

$\sigma = \theta : g \mapsto (g^t)^{-1}$

$K = SO(2)$



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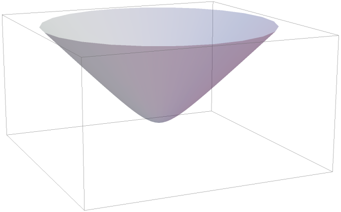
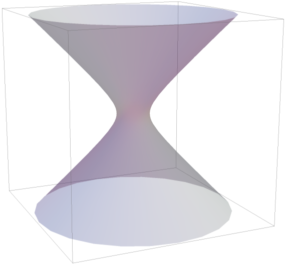


$\sigma : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} d & c \\ b & a \end{pmatrix}$

$H = SO(1,1)$



Horospherical transform for a reductive symmetric space

$$X \xleftarrow{\pi_X} Z \xrightarrow{\pi_\Xi} \Xi$$

X: reductive symmetric space

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Horospheres

θ Cartan involution commuting with σ
P minimal $\sigma \circ \theta$-stable parabolic subgroup
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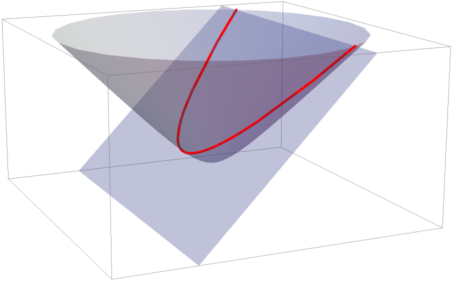
Horosphere

Orbit of a conjugate subgroup of N_P of maximal dimension ($\dim N_P$),
i.e., submanifold of the form $g_1 N_P g_2 \cdot x_0$ of max. dim.
($g_{1,2} \in G, x_0 = e \cdot H$)

Examples

Upper sheet of the hyperboloid

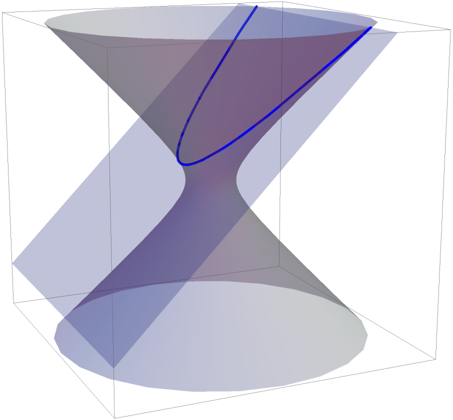


η in the forward light cone

$\langle \eta, x \rangle = p$

Examples

Hyperboloid of one sheet



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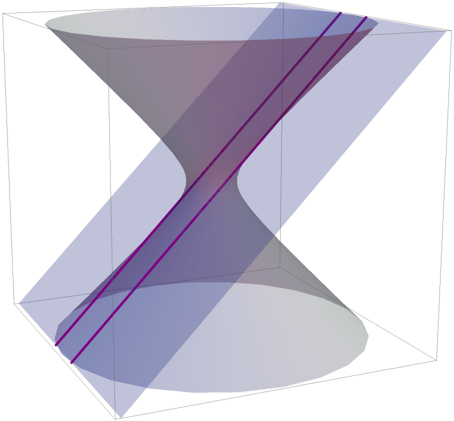

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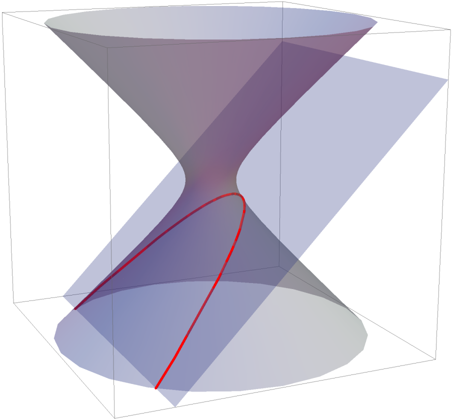

η in the forward light cone

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Horospherical transform

$$Z = \mathrm{diag}(G) \cdot (x_0, \xi_P) \simeq G/(L_P \cap H)$$

$$G/H = X \qquad\qquad \Xi_P = G/(L_P \cap H)N_P$$

$$x_0 = e \cdot H, \ \xi_P = e \cdot (L_P \cap H)N_P$$

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Radon transforms

$\mathcal{R}_P : C_0^\infty(X) \to C^\infty(\Xi_P)$

$$\mathcal{R}_P\phi(g \cdot \xi_P) = \int_{N_P} \phi(gn \cdot x_0)\, dn$$

$\mathcal{R}_P^* : C_0^\infty(\Xi_P) \to C^\infty(X)$

$$\mathcal{R}_P^*\psi(g \cdot x_0) = \int_{H/(L_P \cap H)} \psi(gh \cdot \xi_P)\, dh$$

Polar decomposition

eigenspace decomposition for θ: $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
eigenspace decomposition for σ: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$

Let $\mathfrak{a}_\mathfrak{q} = \mathcal{Z}\big(\mathrm{Lie}(L_P)\big) \cap \mathfrak{p} \cap \mathfrak{q}$

The maps

$$K \times \mathfrak{a}_\mathfrak{q} \to X; \qquad (k, Y) \mapsto k\exp(Y) \cdot x_0,$$
$$K \times \mathfrak{a}_\mathfrak{q} \to \Xi_P; \qquad (k, Y) \mapsto k\exp(Y) \cdot \xi_P$$

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For $B \subseteq \mathfrak{a}_\mathfrak{q}$, define

$$X(B) = K \exp(B) \cdot x_0, \quad \Xi_P(B) = K \exp(B) \cdot \xi_P.$$

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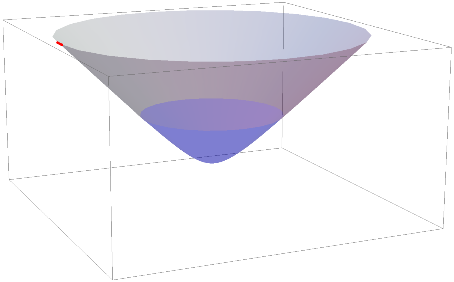
Example

$$K = SO(2), \quad \mathfrak{a}_\mathfrak{q} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\}$$

Support of a transformed function

Let $B \subset \mathfrak{a}_\mathfrak{q}$ convex, compact

$\phi \in C_0^\infty(X)$ s.t. $\text{supp}(\phi) \subseteq X(B)$



$$\exp\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \cdot \xi_P$$

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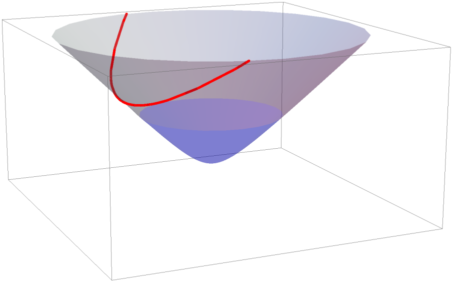

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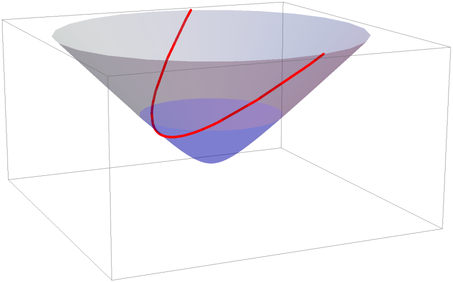

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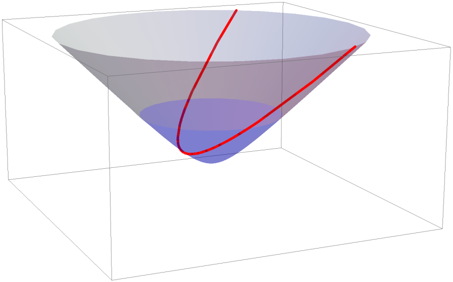

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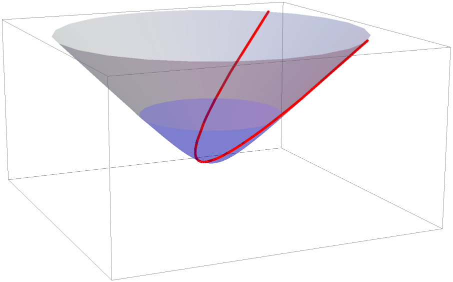

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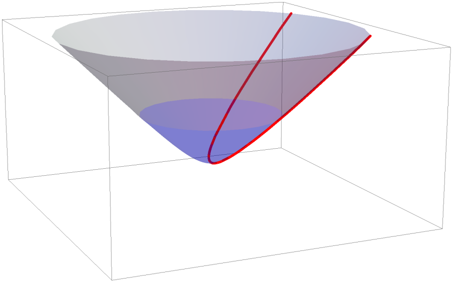

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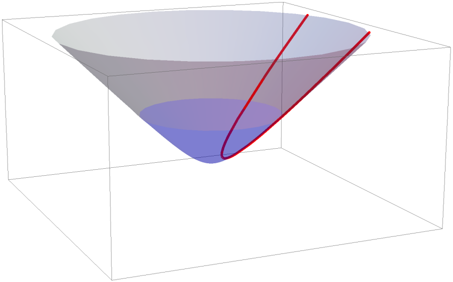

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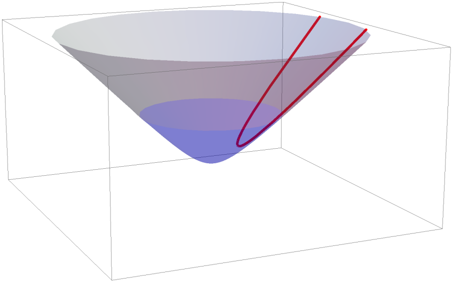

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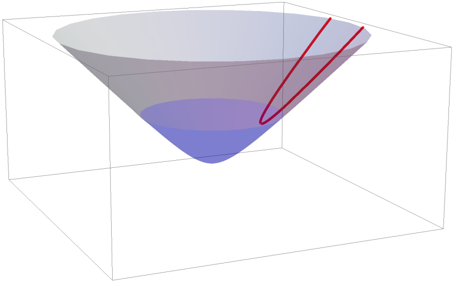

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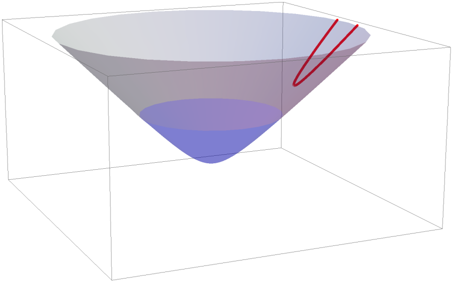

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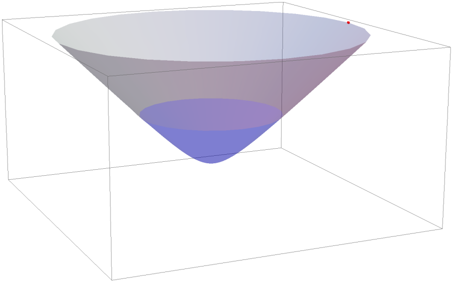

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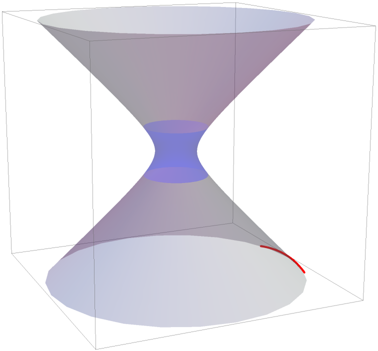

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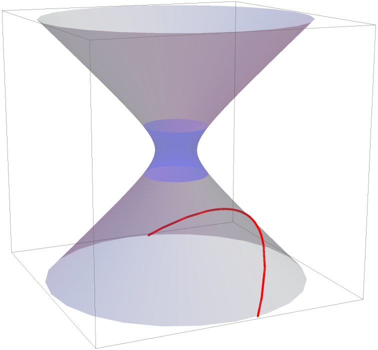

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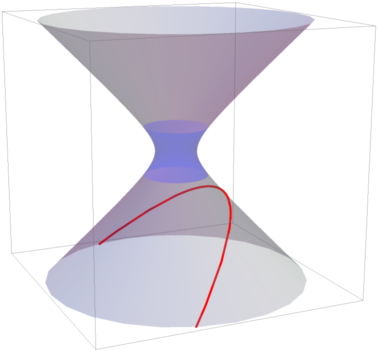

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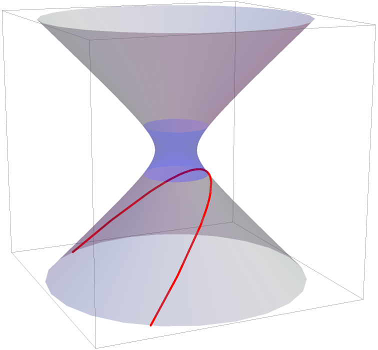

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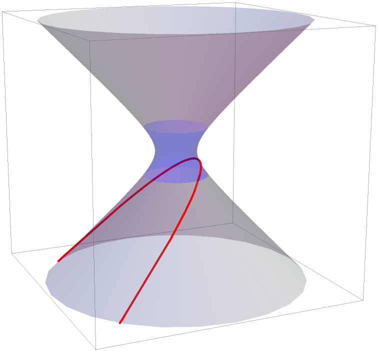

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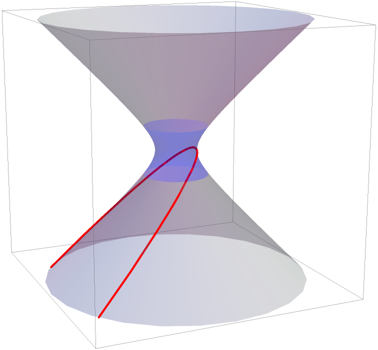

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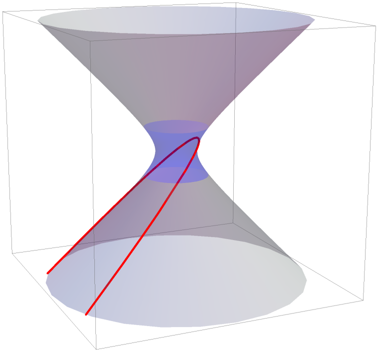

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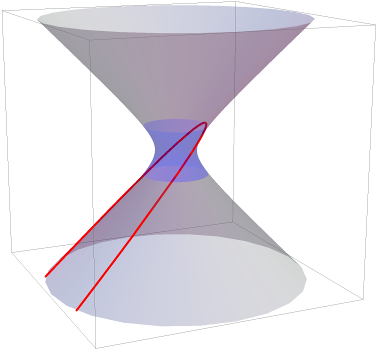

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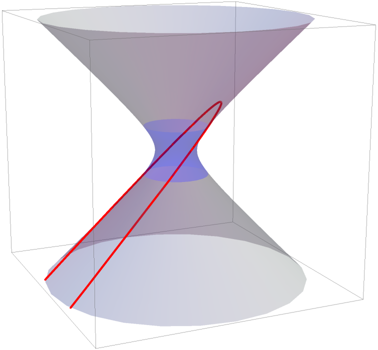

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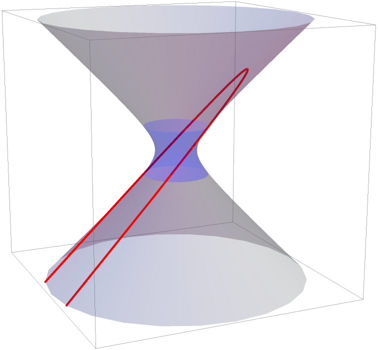

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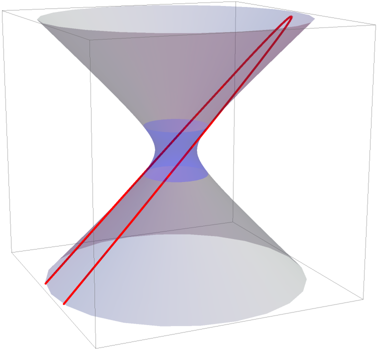

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Cones

Let

$$\Sigma^{+} = \Sigma^{+}(\mathfrak{a}_{\mathfrak{q}}; P)$$
$$\Sigma_{-}^{+} = \Sigma_{-}^{+}(\mathfrak{a}_{\mathfrak{q}}; P) = \{\alpha \in \Sigma^{+}(\mathfrak{a}_{\mathfrak{q}}; P) : (1 - \sigma \circ \theta)\mathfrak{g}_{\alpha} \neq 0\}.$$

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Definition

$$\Gamma_P^- = \sum_{\alpha \in \Sigma^+_-} \mathbb{R}_{\geq 0} H_\alpha, \qquad \Gamma_P = \sum_{\alpha \in \Sigma^+} \mathbb{R}_{\geq 0} H_\alpha$$

Cones

Let

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N.B.: If X is a Riemannian symmetric space, then $\Gamma_P^- = 0$.

Support

Support of a transformed function

Let $B \subset \mathfrak{a}_\mathfrak{q}$ convex, compact, $\phi \in C_0^\infty(X)$.

If $\mathrm{supp}(\phi) \subseteq X(B)$, then $\mathrm{supp}(\mathcal{R}_P\phi) \subseteq \Xi_P(B + \Gamma_P^-)$.

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Support theorem

Partial converse

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Support theorem

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If $\mathrm{supp}(\mathcal{R}_P\phi) \subseteq \Xi_P(B + \Gamma_P)$, then $\mathrm{supp}(\phi) \subseteq X(C)$,

where $C = \bigcap_{k \in \mathcal{N}_{K\cap H}(\mathfrak{a}_\mathfrak{q})} k \cdot (B + \Gamma_P)$

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Generalization of the support theorem of Helgason (1973)

Implies injectivity of $\mathcal{R}_P$

Sketch of the proof

Relation to unnormalized Fourier transform $\mathcal{F}_X^{\mathrm{un}}$:

$$\mathrm{pr}_1\, \mathcal{F}_X^{\mathrm{un}} \sim \mathcal{F}_{cpt.hom.} \circ \mathcal{F}_{\mathfrak{a}_\mathfrak{q}} \circ \mathcal{R}_P$$

Paley-Wiener estimate for $\mathrm{pr}_1\, \mathcal{F}_X^{\mathrm{un}}$

Paley-Wiener estimate for 1 component of the normalized Fourier transform

Inversion formula

Equivariance

Happy Birthday!