Today's lecture : An overview of semi-classical analysis

The underlying theme in this subject : the Bohr correspondence principle :

"Classical mechanics is the $\hbar \to 0$ limit of quantum mechanics."

Classical mechanics for us :

<u>The symplectic category.</u>

Definitions

1. A symplectic manifold, M, is a pair (M^{2n}, ω) where $\omega \in \Omega^2(M)$, $d\omega = 0$ and $\omega_p^n \neq 0$

2. $M^- = (M, -\omega)$

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3. An n dimensional submanifold, Λ, of M is Lagrangian if $i_\Lambda^* \omega = 0$.

4. Let M_1 and M_2 be symplectic manifolds. A canonical relation is a Lagrangian submanifold, Γ, of $M_1^- \times M_2$.

5. Double arrow notation

$$\Gamma : M_1 \Rightarrow M_2$$

6. The symplectic category

a) The objects in this category:

symplectic manifolds M

b) The morphisms in this category:

canonical relations $\Gamma : M_1 \Rightarrow M_2$

c) The composition law:

Let $\Gamma_i : M_i \Rightarrow M_{i+1} \quad i = 1, 2$

be morphisms. Then

$$\Gamma_2 \circ \Gamma_1 : M_1 \Rightarrow M_3$$

is the set of all pairs, (m_1, m_3) for which there exists an $m_2 \in M_2$ with $(m_i, m_{i+1}) \in \Gamma_i$, $i = 1, 2$.

Warning: Transversality assumption are needed to insure that $\Gamma_2 \circ \Gamma_1$ is a canonical relation

d) Transposes. Given a morphism, $\Gamma : M_1 \Rightarrow M_2$, $\Gamma^t : M_2 \Rightarrow M$ is the morphism $\{ (m_2, m_1), (m_1, m_2) \in \Gamma \}$.

e) Point morphisms: Designating "pt" to be the unique (up to symplectomorphism) connected zero dimensional symplectic manifold, Lagrangian submanifolds of M are point morphisms $pt \Rightarrow M$

The topic of the second half of this lecture : <u>quantization</u>

<u>Remark</u> Henceforth all symplectic manifolds will be cotangent bundles

$$(M, \omega) = (T^*X, \omega)$$

where $\omega = -d\left(\sum \xi_i \, dx_i\right)$

Some examples of Lagrangian manifolds and canonical relations

1. $\varphi \in C^\infty(X)$

$$\Lambda_\varphi = \{ (x,\xi) \in T^*X \,,\ \xi = d\varphi_x \}$$

2. Z a submanifold of X and $\Lambda = N^*Z$ its conormal bundle in T^*X.

3. $f : X_1 \to X_2$ a C^∞ map

$$\Gamma_f : T^*X_1 \Rightarrow T^*X_2$$

$$(x_1, \xi_1, x_2, \xi_2) \in \Gamma_f \quad \text{iff} \quad \begin{cases} x_2 = f(x_1) \\ \xi_1 = df_{x_1}^{\,t}\, \xi_2 \end{cases}$$

4. $\Gamma : T^*X_1 \Rightarrow T^*X_2$ is a canonical relation if the set

$$\Gamma^{\#} = \{ (x_1, -\xi_1, x_2, \xi_2) \,;\, (x_1, \xi_1, x_2, \xi_2) \in \Gamma \}$$

is a Lagrangian submanifold of $T^*(X_1 \times X_2)$

5. <u>Double fibrations</u>

$Z \subseteq X_1 \times X_2$, $\pi_i : Z \to X_i$ a fibration.

Then $\Gamma_{\pi_2}^t \circ \Gamma_{\pi_1} = \Gamma$ where

$$\Gamma^{\#} = N^*Z \text{ in } T^*(X_1 \times X_2)$$

How do we "quantize" these canonical relations?

<u>Lemma</u> If $\Lambda \subseteq T^*X$ is a Lagrangian manifold there exists (locally) a fibration $\pi : Z \to X$ and a $\varphi \in C^\infty(Z)$ such that $\Lambda : pt \Rightarrow M$ is the composition of $\Lambda_\varphi : pt \to T^*Z$ and

$\Gamma_\pi : T^*Z \Rightarrow T^*X$

Let $I(X;\Lambda)$ be the space of "rapidly oscillating" functions on X of the form

$$\pi_* \, a(z,\hbar)\, e^{i\frac{\varphi(z)}{\hbar}} \,, \quad \hbar > 0$$

where $a(z,\hbar) \in C^\infty(Z \times \mathbb{R})$ and π_* is the "fiber integration" operator, i.e the L^2-transpose of the pull-back operator $\pi^* : C^\infty(X) \to C^\infty(Z)$.

<u>Quantization</u>: Given a canonical relation $\Gamma : T^*X_1 \to T^*X_2$ we associate to it the set of operators, $F_\hbar : C_0^\infty(X_1) \to C^\infty(X_2)$ where

$$(F_\hbar f)(x_2) = \int K(x_1, x_2, \hbar)\, f(x_1)\, dx_1$$

and $K \in \underline{I}(X_1 \times X_2, \Gamma^\#)$

<u>Notation</u> We'll denote the space of all such operators by $\mathcal{F}(\Gamma)$.

Why is the quantization functor a functor?

<u>Theorem</u> If $\Gamma_i : T^*X_i \Rightarrow T^*X_{i+1}$, $i = 1, 2$ is a canonical relation and $F_{i,h} \in \mathcal{F}(\Gamma_i)$ then for all $\rho \in C_0^\infty(X_2)$, $F_2 \rho F_1 \in \mathcal{F}(\Gamma_2 \circ \Gamma_1)$

Examples

1. The identity operator on $C^\infty(\mathbb{R}^n)$. Here

$$K(x, y, h) = (2\pi h)^{-n} \int e^{\frac{i(x-y)\cdot\xi}{h}} d\xi$$

2. Semi-classical differential operators on $\mathbb{R}^n$ of the form

$$\sum a_\alpha(x, \hbar) \left(\frac{1}{i} \hbar \frac{\partial}{\partial x}\right)^\alpha .$$

Here

$$K(x, y, \hbar) = (2\pi h)^{-n} \int \left(\sum a_\alpha \xi^\alpha\right) e^{\frac{i(x-y)\cdot\xi}{\hbar}} d\xi$$

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3. Let X be a manifold

and $f : X \longrightarrow \mathbb{R}^n$ a C^∞ map

Then the pull back operator

$$f^* : C^\infty(\mathbb{R}^n) \to C^\infty(X)$$

has Schwartz Kernel

$$K_f(x, y, \hbar) = \left(\frac{1}{2\pi\hbar}\right)^n \int e^{i\frac{(f(x)-y)\cdot\xi}{\hbar}}\, d\xi$$

In examples 1 and 2 the underlying canonical relation is the identity map :

$$T^*\mathbb{R}^n \to T^*\mathbb{R}^n$$

and in example 3 the canonical relation $\Gamma_f : T^*X \Rightarrow T^*\mathbb{R}^n$

4. Integral operators

Let $Z \subseteq X_1 \times X_2$ and assume

$\pi_i : Z \to X_i$, $i = 1, 2$ are fibrations.

Then $(\pi_2)_* \pi_1^* : C_0^\infty(X_1) \to C^\infty(X_2)$

is in $\mathcal{F}(\Gamma)$ where $\Gamma = \Gamma_{\pi_2}^t \circ \Gamma_{\pi_1}$

i.e as we saw above

$$\Gamma^\# = N^* Z \text{ in } T^*(X_1 \times X_2)$$

How does the BCP play a role in this subject? In some sense the main goal of semi-classical analysis is validating the BCP: relating operator theoretical properties of operators in the spaces $\mathcal{I}(\Gamma)$ to functorial properties of the Γ's themselves as $\hbar \to 0$

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An example. Let

$$\Gamma_i : T^*X_i \to T^*X_{i+1}$$

$i = 1, 2$ be canonical relations

and $F_{i,h} \in \mathcal{F}(\Gamma_i)$. Then if

$$\Gamma_2 \circ \Gamma_1 = \emptyset,$$

$$F_{2,h}\, \rho\, F_{1,h} = O(\hbar^\infty)$$

for all $\rho \in C_0^\infty(X_2)$

We will conclude by pointing out some tie-ins between semi-classical analysis and Lie theory. Let G be a compact connected semi-simple Lie group, (M, ω) a symplectic manifold and $G \times M \to M$ a Hamiltonian action of G on M.

1. The moment map. From this action we get an infinitesimal action of $\mathfrak{g}$

$$v \in \mathfrak{g} \to v_M \in \mathrm{Vect}(M)$$

and a moment map

$$\Phi : M \to \mathfrak{g}^*$$

defined by the recipe

$$d\langle \Phi, v\rangle = -\iota(v_M)\omega$$

2. Coadjoint orbits.

Let $\mathcal{O}$ be an orbit of the coadjoint action of G on $\mathfrak{g}^*$. Then there exists a unique symplectic form $\omega_{\mathcal{O}}$ on $\mathcal{O}$ with moment map $\mathcal{Z}_{\mathcal{O}} : \mathcal{O} \to \mathfrak{g}^*$

3. Symplectic reduction

Let M be a Hamiltonian G manifold with moment map $\phi : M \to \mathfrak{g}^*$. Suppose G acts freely on the zero level set

$$Z = \phi^{-1}(0)$$

of ϕ. Let π be the projection of Z onto the quotient, $B = Z/G$.

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<u>Theorem</u> There exists a symplectic form ω_B on B satisfying $\pi^*\omega_B = \iota_Z^*\omega$.

<u>Corollary</u> Let $\Gamma_{red} \subseteq M \times B$ be the image of the map,

$$Z \to M \times B, \quad z \to (z, \pi(z))$$

Then Γ_{red} is a canonical relation:

$$\Gamma_{red} : M \Rightarrow B$$

the "reduction morphism".

4. Applying item 3 to the product symplectic manifold $M \times T^*G$ the "B" above is just M itself so the reduction morphism becomes a canonical relation

$$\Gamma_M : M \times T^*G \Rightarrow M$$

or rearranging factors

$$\Gamma_M : M^- \times M \Rightarrow T^*G$$

5. In item 4 let $M = \mathcal{O}$. Then by composing the morphism

$$\Gamma_{\mathcal{O}} : \mathcal{O}^{-} \times \mathcal{O} \rightarrow T^*G$$

with the morphism, $\Delta_{\mathcal{O}} : pt \rightarrow \mathcal{O}^{-} \times \mathcal{O}$

one gets a morphism

$$\Gamma_{\mathcal{O}} \circ \Delta_{\mathcal{O}} : pt \rightarrow T^*G$$

i.e. a Lagrangian submanifold, $\Lambda_{\mathcal{O}}$ of T^*G, the "character" Lagrangian

The "quantization" of the canonical relations described above are closely tied in with interesting problems and results in representation theory, for instance the quantization of the canonical relation in item 3 with the "quantization commutes with reduction" conjecture

An example of this involving item 5: Let T be the Cartan subgroup of G, $\alpha \in \mathfrak{t}_+^*$ a positive weight, $\mathcal{O}$ the coadjoint orbit in $\mathfrak{g}^*$ containing α and χ_α the character of the irreducible representation of G with highest weight α

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Semi-classically this is not a very interesting function. Much more interesting is the character associated with the "semi-classical" weight $\frac{\alpha}{\hbar}$. For this semi-classical weight to be well-defined we have to impose the integrality condition $\frac{1}{\hbar} \in \mathbb{Z}_+$ on $\hbar$

However if we do so we get

<u>Theorem</u> $\gamma_{\frac{e}{h}} \in I^{0}(G, \Lambda_0)$

Finally I'll conclude by exposing you to a typical "semi-classical" computation:

Let $S_\hbar = -\hbar^2 \Delta_{\mathbb{R}^n} + V(x)$

be the Schroedinger operator

If $V(x) \to +\infty$ as $x \to \infty$

this has discrete spectrum

i.e. Fixing $I = (a,b) \in \mathbb{R}$

there exist a finite number of

eigenvalues : $\lambda_u(\hbar)$ $u = 1, \dots, N(\hbar)$

with $a < \lambda_u(\hbar) < b$.

Let $V_{\lambda_u}(\hbar)$ be the eigenspace

$\leftrightarrow \lambda_u(\hbar)$ and $V_{I,\hbar} = \bigoplus V_{\lambda_u}(\hbar)$

Now suppose $G \subseteq O(n)$ and

$V \in C^\infty(\mathbb{R}^n)^G$

Then one gets a representation of G on $V_{I,\hbar}$. Let $N_I(\alpha, h)$ be the number of times the irreducible representation with highest weight $\frac{\alpha}{\hbar}$ occurs in this representation:

Problem: The asymptotics of $N_I(\alpha, \hbar)$ as $\hbar \to 0$?

This reduces to computing the asymptotics of

$$\text{trace} \int \tau_g^* f(S_\hbar)\, \gamma_{\frac{\alpha}{\hbar}}(g)\, dg$$

where $f \in C_0^\infty(I)$ and

$\tau : G \to \mathrm{Aut}(\mathbb{R}^n)$ is the linear action of G on $\mathbb{R}^n$

But this can be reduced to symbolic computation in the symplectic category involving the composition of the canonical relations

(I) $\Lambda_{O} : pt \Rightarrow T^*G$

(II) $\Gamma^t_M : T^*G \Rightarrow M^- \times M^-$

and

(III) $\Delta^t_M : M^- \times M \Rightarrow pt$

where $M = T^*\mathbb{R}^n$

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What one gets from this symbolic computation: A Weyl law

$$N_I(h, \alpha) \sim C_\alpha \hbar^{-d}$$

where $2d = \dim\left(\left(M \times O\right)_{red}\right)$