Local injectivity for weighted Radon transforms

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Let $m(x, \xi, \eta)$ be a smooth, positive function defined in a neighborhood of $(0, 0, 0) \in \mathbf{R}^3$. For $f(x, y)$ supported in $y \geq x^2$ set

$$R_m f(\xi, \eta) = \int f(x, \xi x + \eta) m(x, \xi, \eta) dx, \quad (\xi, \eta) \approx (0, 0).$$

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Question. For which $m(x, \xi, \eta)$ is it true that

$$R_m f(\xi, \eta) = 0 \quad \text{for } (\xi, \eta) \approx (0, 0)$$

implies

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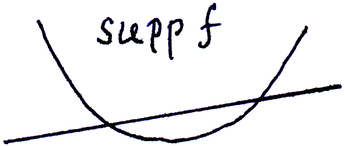

Case $m(x,\xi,\eta) = 1$: classical Radon transform.

Theorem (Strichartz 1982). Assume $f(x,y) = 0$ for $y < x^2$ and

$$Rf(\xi,\eta) = \int f(x,\xi x + \eta)dx = 0, \quad (\xi,\eta) \approx (0,0).$$

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This implies a special case of Helgason's support theorem from 1965:

Let K be compact, convex $\subset \mathbf{R}^n$. Assume $Rf(L) := \int_L f\, ds = 0$ for all hyperplanes L not intersecting K. Then $f = 0$ outside K.

Proof of Strichartz' theorem. Set

$$G_k(\xi, \eta) = \int x^k f(x, \xi x + \eta) dx, \quad k = 0, 1, \ldots .$$

The assumption means that $G_0(\xi, \eta) = 0$. Hence

$$\partial_\xi G_0(\xi, \eta) = \int x \, f'_y(x, \xi x + \eta) dx = 0.$$

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The last expression is equal to $\partial_\eta G_1(\xi,\eta)$, so we have also

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But $G_1(\xi,\eta) = 0$ if the line $y = \xi x + \eta$ does not meet the support of f, that is, if $\eta < -\xi^2/4$. Hence $G_1(\xi,\eta) = 0$ in a nbh of $(0,0)$.

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$$\partial_\xi G_k(\xi,\eta) = \partial_\eta G_{k+1}(\xi,\eta) \quad \text{for all } k.$$

So we obtain $G_k(\xi, \eta) = 0$ for all k in a fixed neighborhood of the origin, which means in particular that

$$G_k(0, \eta) = \int x^k f(x, \eta) dx = 0 \quad \text{for all } k, \eta < \delta,$$

hence $f(x, \eta) = 0$ for all x and all η in a nbh of 0, which completes the proof.

Theorem. R_m is locally injective if $m(x, \xi, \eta)$ is real analytic and positive (JB and Quinto, 1987).

Theorem (JB 1993). There exists a compactly supported function f in the plane, not identically zero, and a positive, smooth weight function $m_L(x, y)$ such that

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Theorem (JB 2010). The set of $m(x, \xi, \eta)$ for which R_m is *not locally injective* is dense in the set of smooth, positive weight functions.

On the other hand, it is well known that the set of weight functions m for which R_m is *globally* injective is open in the C^1 topology.

Proposition. There exists $m(x, \xi, \eta) \in C^\infty$, positive, and $f \in C^\infty$ such that $\operatorname{supp} f \subset \{|x| \leq y\}$, $(0, 0) \in \operatorname{supp} f$, and

$$\int f(x, \xi x + \eta) m(x, \xi, \eta) dx = 0 \quad \text{for } |\xi| < 1/2, \eta < 1.$$

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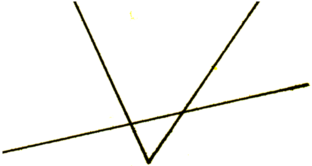

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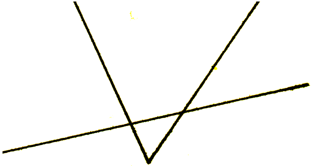


Assume $f(t)$ is a continuous function that changes sign on $[a, b]$. Then there exists a smooth, positive function $m(t)$ such that $\int_a^b f(t) m(t) dt = 0$.

Proof idea. Choose $f(x, y)$ first, then define $m(x, \xi, \eta)$ by

$$m(x, y, L) = 1 - c(L)f(x, y), \quad (x, y) \in L.$$

Then

$$R_m f(L) = \int_L f(1 - c_L f)dx = \int_L f\,dx - c(L)\int_L f^2 dx,$$

so $R_m f = 0$ iff

$$c(L) = \int_L f\,dx \Big/ \int_L f^2 dx.$$

If $\int_L f\,dx$ is sufficiently small, then $m > 0$.

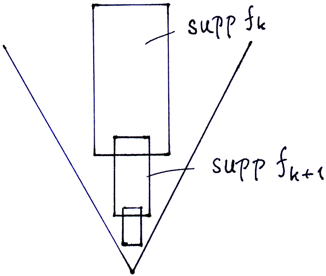
Choose $\qquad f_k(x,y) = \phi(2^k x, 2^k y)\cos 4^k x,$

where $\phi \in C^\infty$, is supported in the rectangle

$$|x| \le 1, \quad 1 \le y \le 3,$$

$\phi = 1$ in a slighly smaller rectangle, and

$$f(x,y) = \sum_{k=1}^{\infty} f_k(x,y)/k!.$$



The following is obvious:

$$f \in C^\infty, \qquad \operatorname{supp} f \subset \{|x| \leq y\}$$

$$\int_{L(\xi,\eta)} f^2 dx > 0 \quad \text{if } \eta > 0.$$

Hence

$$c(L) = C(L(\xi, \eta)) = \int_L f dx \Big/ \int_L f^2 dx \in C^\infty, \quad \eta > 0.$$

Lemma. For every p there exists C_p such that

$$\left|\partial^{\alpha}_{(\xi,\eta)} \int_{L(\xi,\eta)} f_k dx\right| \leq C_p 2^{-kp}, \quad |\alpha| \leq p.$$

Note: if L intersects supp f_k and $|\xi| < 1/2$, then $\eta \sim 2^{-k}$.

We can make the $c(L)$ smaller by introducing a parameter λ in the definition of f_k,

$$f_k(x,y) = \phi(2^k x, 2^k y) \cos 4^k \lambda x,$$

and choosing λ sufficiently large. This will make

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as close to 1 as we like. Moreover, starting with an arbitrary $m_0(x, y, L)$ and choosing

$$m(x, y, L) = m_0(x, y, L) - c(L) f(x, y),$$

$$c(L) = \int_L f m_0 \, dx \Big/ \int f^2 dx$$

we can make $m(x, y, L)$ as close as we wish to an arbitrary given $m_0(x, y, L)$.

The attenuated Radon transform

Denote for a moment points in the plane by $\mathbf{x} = (x_1, x_2)$ instead of (x, y). The attenuation weights can then be written

$$m(\mathbf{x}, L) = \exp\left(-\int_{L_{\mathbf{x}}} a(\cdot)ds \right), \quad \mathbf{x} \in L,$$

for some compactly supported function $a(\mathbf{x})$ on the plane. Here $L_{\mathbf{x}}$ is one of the segments of the line L, divided at the point $\mathbf{x}$.

In the familiar coordinates (ω, p), where $L(\omega, p)$ is the line $\mathbf{x} \cdot \omega = \omega_1 x_1 + \omega_2 x_2 = p$, the attenuation weight can be written

$$m(\mathbf{x}, \omega) = \exp\left(-\int_0^\infty a(\mathbf{x} + t\omega^\perp)dt \right). \qquad (1)$$

Inversion formulas for this class of $m(\mathbf{x}, p)$ were proved by:

Arbuzov, Bukhgeim, and Kazantsev 1998
Novikov 2002
Natterer 2001
Boman and Strömberg 2004
Bal 2004
Fokas and Sung 2005

JB and Strömberg proved an inversion formula under the more general condition that $m(\mathbf{x}, \omega)$ is a product of a function of type (1) and one of the type

$$m(\mathbf{x}, \omega) = \exp\left(\int_0^\infty \langle \omega^\perp, b(\mathbf{x} + t\omega^\perp) \rangle dt \right),$$

where $b(\mathbf{x}) = (b_1(\mathbf{x}), b_2(\mathbf{x}))$ is a vector field on the plane. Equivalently

$$\langle \omega^\perp, \partial_{\mathbf{x}} \rangle \log m(\mathbf{x}, \omega) = a(\mathbf{x}) + \omega_1 b_1(\mathbf{x}) + \omega_2 b_2(\mathbf{x}).$$

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If $a(\mathbf{x}) = 0$, then $m(\mathbf{x}, \omega)$ is ω-even, $m(\mathbf{x}, -\omega) = m(\mathbf{x}, \omega)$, up to a trivial factor, and if $b_1(\mathbf{x}) = b_2(\mathbf{x}) = 0$, then $\log m(\mathbf{x}, \omega)$ is odd up to a trivial term.

By a trivial factor I mean a factor $q(\mathbf{x}, L) = c(L)$ that depends only on the line L.

For instance, if $b_1(\mathbf{x}) = b_2(\mathbf{x}) = 0$ we can write

$$\log m(\mathbf{x}, L) = \int_{L_{\mathbf{x}}} a(\cdot)ds - \int_L a(\cdot)ds = -\int_{L'_{\mathbf{x}}} a(\cdot)ds,$$

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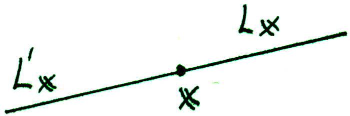

With affine coordinates, $y = \xi x + \eta$, we have $\omega = (\xi, -1)$, $\omega^\perp = (1, \xi)$, and the condition on $m(x, y, \xi)$ in the case $a = 0$ can be written

$$(\partial_x + \xi \partial_y) \log m(x, y, \xi) = b_1(x, y) + \xi b_2(x, y). \qquad (2)$$

This condition is invariant under projective transformations.

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Projective transformations have the property

$$\frac{ds_{\widetilde{L}}}{ds_L} = \alpha(\mathbf{x})\beta(L).$$

Theorem (Gindikin 2009). Assume that $m(x, \xi, \eta)$ satisfies

$$\big(\partial_\xi - x\,\partial_\eta\big) \log m(x, \xi, \eta) = x\, a(\xi, \eta) + b(\xi, \eta). \qquad (3)$$

for some functions $a(\xi, \eta)$ and $b(\xi, \eta)$. Then

$$f(0, 0) = c \int \int \frac{(\partial_\eta + a(\xi, \eta))R_m(\xi, \eta)}{m(0, \xi, \eta)} \, \frac{d\eta}{\eta} d\xi.$$

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The operator $\partial_\xi - x\partial_\eta$ appearing in Gindikin's condition is the directional derivative in the direction of rotating the line through a fixed point (x,y). Because

$$\partial_\xi\left(m(x,\xi,y-\xi x)\right) = m'_\xi(x,\xi,y-\xi x) - x\,m'_\eta(x,\xi,y-\xi x).$$

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We saw above that if $m(x, \xi, \eta) = 1$, then

$$\partial_\xi G_k = \partial_\eta G_{k+1} \quad \text{for all } k.$$

In this case we have instead

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Arguing as before we can use this fact to prove that $G_k(\xi, \eta) = 0$ for all k in a fixed nbh of the origin, which implies the assertion.

Given a 2-parametric curve family

$$y = u(x, \xi, \eta) \approx \eta + \xi x + \mathcal{O}(x^2)$$

and a weight function $m(x, \xi, \eta)$, set

$$Rf(\xi, \eta) = R_{u,m} f(\xi, \eta) = \int f(x, u(x, \xi, \eta)) m(x, \xi, \eta) dx, \qquad (4)$$

and ask the same question as before.

To define R invariantly, denote (x, y)-space by X and the space of curves by Γ. Then the equation $y = u(x, \xi, \eta)$ defines a hypersurface Z in the product space $X \times \Gamma$. The hypersurface Z is the incidence relation:

$$\{(\mathbf{x}, \gamma);\ \mathbf{x} \in \gamma\}.$$

Using the projections

$$\pi_X : \ (\mathbf{x}, \gamma) \mapsto \mathbf{x}, \qquad \pi_\Gamma : \ (\mathbf{x}, \gamma) \mapsto \gamma,$$

$$\begin{array}{ccc} & Z & \\ {}^{\pi_X}\swarrow & & \searrow^{\pi_\Gamma} \\ X & & \Gamma \end{array}$$

we can define two kinds of fibers on Z:

$$\pi_\Gamma^{-1}(\gamma) \qquad \text{and} \quad \pi_X^{-1}(\mathbf{x}).$$

and projecting those down to X and Γ we obtain

$$\pi_X(\pi_\Gamma^{-1}(\gamma)) \qquad \text{and} \quad \pi_\Gamma(\pi_X^{-1}(\mathbf{x})).$$

If μ is a measure on Z and $f \in C(X)$, $\varphi \in C_c(\Gamma)$ we can form

$$\langle \mu, (f \circ \pi_X)(\varphi \circ \pi_\Gamma) \rangle = \langle Rf, \varphi \rangle = \langle f, R^* \varphi \rangle.$$

If $\mu = m(x, \xi, \eta) dx d\xi d\eta$ we get back the expression (4) for Rf.

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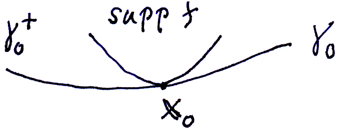
Question. For which curve families and densities is it true that unique continuation across γ_0 holds near $\mathbf{x}_0$ for solutions of $Rf = 0$ in the sense that

$$\begin{aligned} \operatorname{supp} f \subset \gamma_0^+ \cup \{\mathbf{x}_0\} \quad \text{and} \quad Rf = 0 \text{ near } \mathbf{x}_0 \\ \Longrightarrow \quad f = 0 \quad \text{near } \mathbf{x}_0. \end{aligned}$$



The condition on $\mu = m(x, \xi, \eta) dx d\xi d\eta$ can be invariantly expressed as follows:

$$\text{there exists a } 1\text{-form } \sigma \text{ on } \Gamma \text{ such that}$$
$$V(\mu) = (\pi_\Gamma^*(\sigma) \llcorner V)\mu \quad \text{for all vector fields } V \tag{5}$$
$$\text{that are tangent to the fibers } \pi_X^{-1}(\mathbf{x}).$$

The operation of a vector field on a density is defined by $\langle V(\mu), \varphi \rangle = -\langle \mu, V(\varphi) \rangle$.

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In the special case when Γ is the manifold of lines $y = \xi x + \eta$, condition (5) means the same as (3), that is

$$(\partial_\xi - x\partial_\eta)\, m(\xi,\eta,x) = \big(a(\xi,\eta) + x\, b(\xi,\eta)\big)\, m(\xi,\eta,x).$$

To see this, choose $\sigma = a(\xi, \eta)d\xi + b(\xi, \eta)d\eta$.
The fiber $\pi_X^{-1}(\mathbf{x}) = \pi_X^{-1}(x, y)$ is

$$\{(\xi, \eta, x);\ \xi x + \eta = y\}.$$

The vector field $V = \partial_\xi - x\partial_\eta$ is tangent to those fibers.
By definition $V(\mu) = \big(\partial_\xi - x\partial_\eta\big) m(\xi, \eta, x) dx d\xi d\eta$.
Moreover $\quad \pi_\Gamma^*(\sigma) \llcorner V = a(\xi, \eta) - x\, b(\xi, \eta)$,
which proves the claim.

To formulate the condition on Z we solve η from the equation $y = u(\xi, \eta, x)$, obtaining $\eta = \rho(\xi, x, y)$. The condition on Z is as follows: the functions $\xi \mapsto \eta = \rho(\xi, x, y)$ are solutions of an ODE (with x, y as parameters)

$$\eta'' = \Psi(\xi, \eta, \eta'), \tag{6}$$

where $\Psi(\xi, \eta, p)$ is a polynomial in p of degree at most 3.

Theorem. R is locally injective if (5) and (6) are satisfied.

A 2-parametric family of curves in the plane can be defined by a second order differential equation

$$y'' = \Phi(x, y, p), \quad p = dy/dx. \tag{7}$$

Proposition. The class of differential equations (7) for which $p \mapsto \Phi(x, y, p)$ is a polynomial of degree ≤ 3 is invariant under smooth coordinate transformations in the plane.

See Arnold, Geometric aspects of the theory of ODE, chapter 1, §6.

The condition used in the proof is

$$\rho''_{\xi\xi} = a_0(\rho'_\xi)^2 + a_1\rho'_\xi + a_2$$

for some functions a_j that are constant on fibers $\pi_\Gamma^{-1}(\gamma)$, that is

$$a_j(x, y, \xi) = A_j(\xi, y - \xi x)$$

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Note: the set of curves γ passing through (x, y) forms a curve in (ξ, η)-space, defined by $\eta = \rho(\xi, x, y)$.

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