Cusp Forms on Hyperbolic Spaces

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joint work with Mogens Flensted-Jensen
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Abstract

Cusp forms on a group G can be defined as the kernel of certain Radon transforms on G. Cusp Forms on real reductive Lie groups G were introduced by Harish-Chandra, who also showed that they coincide precisely with the discrete part of the spectral decomposition of the space of square integrable functions on G.

Flensted-Jensen recently proposed a new family of Radon transforms and associated Cusp forms on Reductive Symmetric Spaces, which in the group case reduces to the definition of Harish-Chandra. We will in this talk discuss Cusp Forms on Hyperbolic Spaces, in particular the existence of non-cuspidal discrete series.

History

Flensted-Jensen: Lectures at Oberwolfach, 2001.

Flensted-Jensen: Talk at 24th Nordic and 1st Franco-Nordic Congress of Mathematicians, January 2005, Reykjavik, Iceland.

Andersen: Talk at International Conference on Integral Geometry, Harmonic Analysis and Representation (in honor of Sigurdur Helgason's 80th birthday), August 2007, Reykjavik, Iceland.

Schlichtkrull: Oberwolfach report, 2007.

Work by van den Ban and Kuit; and by van den Ban, Kuit and Schlichtkrull.

Article on arXiv: http://lanl.arxiv.org/abs/1111.4031

Real Hyperbolic Spaces

$G = SO(p, q+1)_e$. (for talk assume $q > 1$)

$H = SO(p, q)_e$: connected subgroup of G stabilizing $(0, \ldots, 0, 1)$.

The symmetric space G/H identified with real hyperbolic space:

$$X = \{x \in \mathbb{R}^{p+q+1} : x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_{p+q+1}^2 = -1\}.$$

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$\mathfrak{g}$, $\mathfrak{h}$: the Lie algebras of G, H.
θ: the classical Cartan involution on G, $\mathfrak{g}$.
σ: involution fixing H, $\mathfrak{h}$.
$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{h} \oplus \mathfrak{q}$ decomposition of $\mathfrak{g}$ into the ± 1-eigenspaces of θ, σ.
$K = SO(p+1) \times SO(q+1)$: maximal compact subgroup (fixed by θ), with Lie algebra $\mathfrak{k}$.

Polar decomposition

$\mathfrak{a}_q \subset \mathfrak{p} \cap \mathfrak{q}$ maximal abelian subalgebra:

$$\mathfrak{a}_\mathfrak{q} = \left\{ X_t = \begin{pmatrix} 0 & 0 & t \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Corresponding abelian subgroup $A = \{a_t\} \subset G$:

$$a_t = \exp(X_t) = \begin{pmatrix} \cosh t & 0 & \sinh t \\ 0 & I_{p+q-1} & 0 \\ \sinh t & 0 & \cosh t \end{pmatrix}.$$

Cartan decomposition $G = KAH$ gives *polar coordinates* on X:

$$K \times \mathbb{R} \ni (k, t) \mapsto k a_t H \in X.$$

Discrete series: Parametrization

Let $\rho = \frac{1}{2}(p+q-1)$, $\rho_c = \frac{1}{2}(q-1)$ and $\mu_\lambda = \lambda + \rho - 2\rho_c$.

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Descends to the projective space if and only if μ_λ is even.

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$$\xi_\lambda(k_\theta a_t) = \cos\theta\ P_\lambda(\cosh^2 t)(\cosh t)^{-\lambda-\rho-2m}.$$

(P_λ polynomial of degree m).

Nilpotent subgroups

Sum of positive root spaces, with $u \in \mathbb{R}^{p-1}$ and $v \in \mathbb{R}^q$ (row) vectors:

$$\mathfrak{n} = \begin{pmatrix} 0 & u & v & 0 \\ -u^t & 0 & 0 & u^t \\ v^t & 0 & 0 & -v^t \\ 0 & u & v & 0 \end{pmatrix}.$$

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$N = \exp(\mathfrak{n})$ is too BIG!

$\int_N f(n)\, dn$ diverges for $f(ka_tH) = (\cosh t)^{-\rho-\nu} \in \mathcal{C}(X)$ when $0 < \nu \leq \frac{1}{2}(p+q-3)$.

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$$\mathfrak{n}^* = \text{"}\{u \in \mathbb{R}^{p-1}, v \in \mathbb{R}^q, u_j = v_j \text{ for } j = 1, \ldots, l\}\text{"}.$$

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Limits. Let f be K-invariant, then $\lim_{s\to\infty} e^s Rf(s)$ exists for $p < q$...

Abel transform and differential operators

Define: $\mathcal{A}f(a) = a^{\rho_1} Rf(a)$ ("Abel transform"), where

$$\rho_1 = \begin{cases} \frac{1}{2}(p-q-1) & \text{if } p > q \\ \frac{1}{2}(q-p+1) & \text{if } p \leq q \end{cases}$$

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which implies:

$$Rf(s) = C_1 e^{(-\rho_1+\lambda)s} + C_2 e^{(-\rho_1-\lambda)s}.$$

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1. *If $\mu_\lambda > 0$, then $Rf = 0$.*
2. *If $\mu_\lambda \leq 0$, then $Rf(s) = Ce^{(-\rho_1+\lambda)s}$ $(s \in \mathbb{R})$, for some $C \neq 0$.*

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There exist non-spherical non-cuspidal discrete series if and only if $q > p + 3$. These representations do not descend to discrete series of the real projective hyperbolic space.