A FAST ITERATIVE METHOD FOR INVERSE PROBLEMS WITH INEXACT FORWARD OPERATOR

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Overview

Inverse problems with inexact forward operator

We consider an inverse problem

$$A(f) = g, \quad A : \mathcal{D}(A) \subseteq X \to Y,$$

where the given data g^δ are subject to noise with noise level

$$\|g^\delta - g\| \leq \delta.$$

In addition, we assume that only an inexact version A^η of A is given with inexactness $\eta > 0$, such that

$$\|A^\eta(f) - A(f)\| \leq \eta \cdot \rho \quad \text{for all } f \in B_\rho(0) \subseteq \mathcal{D}(A).$$

$\rightsquigarrow$ **include information on inexactness in reconstruction**

Inexact forward operators: two examples

Dynamic Computerized Tomography → Talk by Bernadette Hahn, Tuesday 9:40

A slight motion Γ of the object during the scan affects the data:

$$g^{\Gamma}(\varphi, s) = R^{\Gamma} f_0(\varphi, s) = \int_{\mathbb{R}^2} f_0(\Gamma_{\varphi} x)\delta(s - x^T\theta(\varphi))\,\mathrm{d}x$$

We use the static model with inexactness:

$$g^{\Gamma}(\varphi, s) = Rf_0(\varphi, s) + \eta(\varphi, s) = \int_{\mathbb{R}^2} f_0(x)\delta(s - x^T\theta(\varphi))\,\mathrm{d}x + \eta(\varphi, s)$$

Magnetic Particle Imaging

The high complexity of the physical model for the system function s suggests to use an inexact model:

$$u(t) = \int_{\Omega} c(x)\big(s(x,t) + \Delta s(x,t)\big)\,\mathrm{d}x = \int_{\Omega} c(x)s(x,t)\,\mathrm{d}x + \eta(t)$$

Overview

Notation and definitions

Let X, Y be real Hilbert spaces and $M_{A(f)=g} := \{f \in X : A(f) = g\}$ the solution set.

Hyperplanes, halfspaces and stripes

Let $u \in X \setminus \{0\}$ and $\alpha, \xi \in \mathbb{R}$, $\xi \geq 0$. We define the (affine) hyperplane

$$H(u, \alpha) := \{f \in X \ : \ \langle u, f \rangle = \alpha\},$$

the halfspace

$$H_{\leq}(u, \alpha) := \{f \in X \ : \ \langle u, f \rangle \leq \alpha\}$$

and the stripe

$$H(u, \alpha, \xi) := \{f \in X \ : \ |\langle u, f \rangle - \alpha| \leq \xi\}.$$

Sequential subspace optimization (SESOP) for linear inverse problems in Hilbert spaces

$$f_{n+1} := f_n - \sum_{i \in I_n} t_{n,i} A^* w_{n,i},$$

I_n a finite index set, $w_{n,i} \in Y$ for all $i \in I_n$, and the parameters $t_n = (t_{n,i})_{i \in I_n}$ minimize

$$h_n(t) := \frac{1}{2} \left\| f_n - \sum_{i \in I_n} t_i A^* w_{n,i} \right\|^2 + \sum_{i \in I_n} t_i \langle w_{n,i}, g \rangle .$$

Lemma [Schöpfer, Schuster, Louis (2008)]

The minimization of $h_n(t)$ is equivalent to computing the metric projection

$$f_{n+1} = P_{H_n}(f_n), \qquad H_n := \bigcap_{i \in I_n} H_{n,i},$$

onto the intersection of hyperplanes

$$H_{n,i} := \left\{ f \in X \ : \ \langle A^* w_{n,i}, f \rangle = \langle w_{n,i}, g \rangle \right\} \supseteq M_{Af=g}.$$

Adaptions of SESOP

Linear inverse problems with noisy data

Use stripes with width $\xi = \xi(\delta)$:

$$H_{n,i} := \left\{ f \in X \ : \ \left| \left\langle A^* w_{n,i}^\delta, f \right\rangle - \left\langle w_{n,i}^\delta, g^\delta \right\rangle \right| \leq \delta \| w_{n,i}^\delta \| \right\} \supseteq M_{Af=g}.$$

Nonlinear inverse problems with noisy data

$$H_{n,i}^\delta := \Big\{ f \in X : \left| \left\langle A'(f_i^\delta)^* w_{n,i}^\delta, f_i^\delta - f \right\rangle - \left\langle w_{n,i}^\delta, A(f_i^\delta) - g^\delta \right\rangle \right| \leq \| w_{n,i}^\delta \| \left(c_{\mathrm{tc}} \left(\| R_i^\delta \| + \delta \right) + \delta \right) \Big\} \supseteq M_{A(f)=g}$$

RESESOP for noisy data and inexact forward operator

Let

$$\|A^\eta(f) - A(f)\| \leq \eta \cdot \rho \quad \text{for all } f \in B_\rho(0) \subseteq \mathcal{D}(A)$$

for $\eta > 0$.

Linear inverse problems with noisy data and inexact forward operator

$$H_{n,i}^{\eta,\delta} := \left\{ f \in X \ : \ \left| \left\langle (A^\eta)^* w_{n,i}^{\eta,\delta}, f \right\rangle - \left\langle w_{n,i}^{\eta,\delta}, g^\delta \right\rangle \right| \leq (\delta + \eta\rho) \|w_{n,i}^{\eta,\delta}\| \right\} \supseteq M_{Af=g}.$$

Nonlinear inverse problems with noisy data and inexact forward operator

$$H_{n,i}^{\eta,\delta} := \Bigg\{ f \in X : \left| \left\langle (A^\eta)'(f_i^{\eta,\delta})^* w_{n,i}^{\eta,\delta}, f_i^{\eta,\delta} - f \right\rangle - \left\langle w_{n,i}^{\eta,\delta}, A^\eta(f_i^{\eta,\delta}) - g^\delta \right\rangle \right|$$
$$\leq \|w_{n,i}^{\eta,\delta}\| \left(c_{\mathrm{tc}} \left(\|w_{n,i}^{\eta,\delta}\| + (\delta + \eta\rho) \right) + (\delta + \eta\rho) \right) \Bigg\} \supseteq M_{A(f)=g}$$

Semi-discrete setting

Let $A_k : X \to Y_k$ be linear bounded operators and

$$A_k f = g_k, \quad \|g_k^\delta - g_k\| \leq \delta, \quad \|A_k - A\| \leq \eta_k.$$

for all $k = 1, ..., K$ as well as

$$n \in I_n \text{ for all } n \in \mathbb{N}, \quad w_{n,i}^{\eta,\delta} := A_{[n]}^\eta f_i^{\eta,\delta} - g_{[n]}^\delta \text{ for all } i \in I_n, n \in \mathbb{N},$$

where $[n] = n \mod K$.

Time-dependent inverse problems

- The index k may refer to different time points $t_k \in [0, T]$.
- Changes in the physical setting may be incorporated in $\eta_k = \eta(t_k)$.
- Examples:
  - Periodic motion in dynamic CT calls for a periodic function $\eta(t)$;
    Reference state at t_k: $\eta(t_k) = 0$
  - Rising temperature in MPI scanner during the scan: increasing $\eta(t)$

RESESOP-Kaczmarz algorithm

Choose a starting value $f_0^{\eta,\delta} = f_0 \in B_\rho(0) \subseteq X$ and constants $\tau_k > 1$, $k = 1, ..., K$.

As long as the discrepancy principle is not yet fulfilled, i.e.,

$$\left\| A_{[n]}^\eta f_n^{\eta,\delta} - g_{[n]}^\delta \right\| > \tau_{[n]} \left(\eta_{[n]}\rho + \delta_{[n]} \right),$$

calculate the new iterate as

$$f_{n+1}^{\eta,\delta} := P_{H_n^{\eta,\delta}}\left(f_n^{\eta,\delta}\right), \quad H_n^{\eta,\delta} := \bigcap_{i \in I_n^{\eta,\delta}} H\left(u_{n,i}^{\eta,\delta}, \alpha_{n,i}^{\eta,\delta}, \xi_{n,i}^{\eta,\delta}\right).$$

Here, we choose $I_n^{\eta,\delta} \subseteq \{0, 1, ..., n\}$ such that $n \in I_n^{\eta,\delta}$ and

$$\begin{aligned} u_{n,i}^{\eta,\delta} &:= \left(A_{[i]}^\eta\right)^* w_{n,i}^{\eta,\delta}, \\ \alpha_{n,i}^{\eta,\delta} &:= \left\langle w_{n,i}^{\eta,\delta,l}, g_{[i]}^\delta \right\rangle, \\ \xi_{n,i}^{\eta,\delta} &:= \left(\eta_{[i]}\rho + \delta_{[i]}\right) \left\| w_{n,i}^{\eta,\delta} \right\|. \end{aligned}$$

Convergence of the SESOP-Kaczmarz algorithm

Now let $\delta = 0$ and $\eta_k = 0$ for all $k = 1, ..., K$.

Theorem [Blanke, Hahn, W. 2019]

Let $\{f_n\}_{n\in\mathbb{N}}$ be the sequence generated by the SESOP-Kaczmarz algorithm with initial value f_0 and

- $I_n \subseteq \{n - N + 1, ..., n\} \cap \mathbb{N}$, $N \in \mathbb{N} \setminus \{0\}$ fixed,
- $n \in I_n$,
- $w_i := w_{n,i} := A_{[i]} f_i - g_{[i]}$ for all $i \in I_n$

for all $n \in \mathbb{N}$. If there is an upper bound for the set of optimization parameters $t_{n,i}$, i.e., $|t_{n,i}| < t$ for all $i \in I_n, n \in \mathbb{N}$, then $\{f_n\}_{n\in\mathbb{N}}$ converges strongly to a solution f of $A_k f = g_k,\ k = 1, ..., K$.

The RESESOP-Kaczmarz algorithm as a regularization method

Let $n \in I_n^{\eta,\delta} \subseteq \{n-N+1, ..., n\} \cap \mathbb{N}$ and $w_{n,i}^{\eta,\delta} := A_{[i]}^{\eta} f_i^{\eta,\delta} - g_{[i]}^{\delta}$ for all $i \in I_n^{\eta,\delta}$ in the RESESOP-Kaczmarz algorithm.

Theorem [Blanke, Hahn, W. 2019]

(a) The RESESOP-Kaczmarz algorithm yields a finite stopping index n_*.

(b) We have $f_n^{\eta,\delta} \to f_n$ for $\eta \to 0$, $\delta \to 0$, where $\{f_n\}_{n\in\mathbb{N}}$ is the sequence of iterates generated by the SESOP-Kaczmarz algorithm.

(c) If $\{f_n\}_{n\in\mathbb{N}}$ converges strongly, then

$$f_{n_*(\eta,\delta)}^{\eta,\delta} \to f \in M_{Af=g}^{\mathrm{sd}} \cap B_\rho(0).$$

Remark

The previous statements also hold for the sequences $\{f_{rK}\}_{r\in\mathbb{N}}$ and $\{f_{rK}^{\eta,\delta}\}_{r\in\mathbb{N}}$ of *full iterates*.

Overview

Dynamic computerized tomography

Motion in computerized tomography

The object undergoes motion during the scan.

Typical examples from medical CT imaging:

- **Breathing:** the entire body expands periodically (global motion)
- **Heartbeat:** a local periodical motion

Problem: If the motion is not taken into account, this causes severe artefacts in the reconstruction and details are no longer visible.

Motion compensation

In general, the motion is unknown or hard to estimate.

$\rightsquigarrow$ interpret the motion of the object as a model inexactness

Dynamic CT

Consider the dynamic model

$$g^\Gamma(\varphi, s) = R^\Gamma f_0(\varphi, s) = \int_{\mathbb{R}^2} f_0(\Gamma_\varphi x)\delta(s - x^T\theta(\varphi))\,\mathrm{d}x.$$

For the inexactness η in

$$\left\| R^\Gamma(\varphi, s) - R(\varphi, s) \right\| \leq \eta(\varphi, s)$$

we have several options.

Inexactness η

- $\eta(\varphi, s) = \bar{\eta}$ with some constant $\bar{\eta} > 0$
- $\eta(\varphi, s) = \eta(\varphi)$: Inexactness depends on the position of the tomograph, i.e., on time
- $\eta(\varphi, s) = \eta(s)$: Inexactness depends on offset and is affected by local behavior of the object
- $\eta(\varphi, s) = \eta(\varphi, s)$: local time-dependent inexactness

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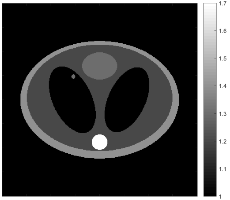
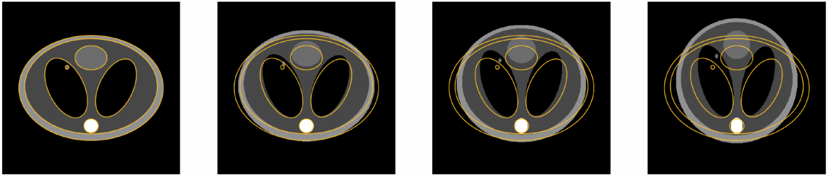
Numerical experiment: affine periodic motion

Reference/initial state:



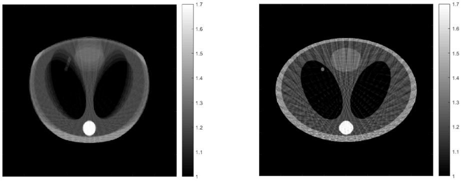
Affine motion:



(a) $j = 1$ (b) $j = 3$ (c) $j = 4$ (d) $j = 6$

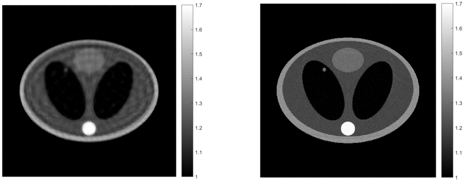
Numerical experiment: affine periodic motion

Reconstructions from **exact data** using the static model:



(a) Full data set (b) Filtered data set

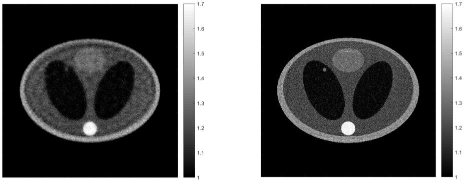
Reconstruction with SESOP-Kaczmarz using exact inexactness (a) $\eta = \eta(\varphi)$ (left) and (b) $\eta = \eta(\varphi, s)$ (right):



Numerical experiment: affine periodic motion

Now we add some equally distributed noise to the data.

Reconstruction with RESESOP-Kaczmarz using exact inexactness (a) $\eta = \eta(\varphi)$ (left) and (b) $\eta = \eta(\varphi, s)$ (right):



Further numerical experiments

- It is possible to use each time point as a reference state, which is then reconstructed by the (RE)SESOP-Kaczmarz algorithm.
- The method also works for non-affine motion.
- An over-estimation of the inexactness η still yields good reconstructions.
- Periodic motions are especially suited for our approach.
- Using a global inexactness does usually not yield good reconstructions.

Overview

Conclusion and outlook

A short summary

- SESOP: Adaption to model inexactness
- SESOP-Kaczmarz algorithm
- Convergence and regularization results
- Application: time dependent problems, dynamic CT

Outlook

- Estimation of inexactness
- Application to data from MicroCT (nondestructive testing)
- Application for other inverse problems (e.g. MPI)

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Some references

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Thank you for your attention!