Karlsruher Institut für Technologie

Seismic imaging with the elliptic Radon transform

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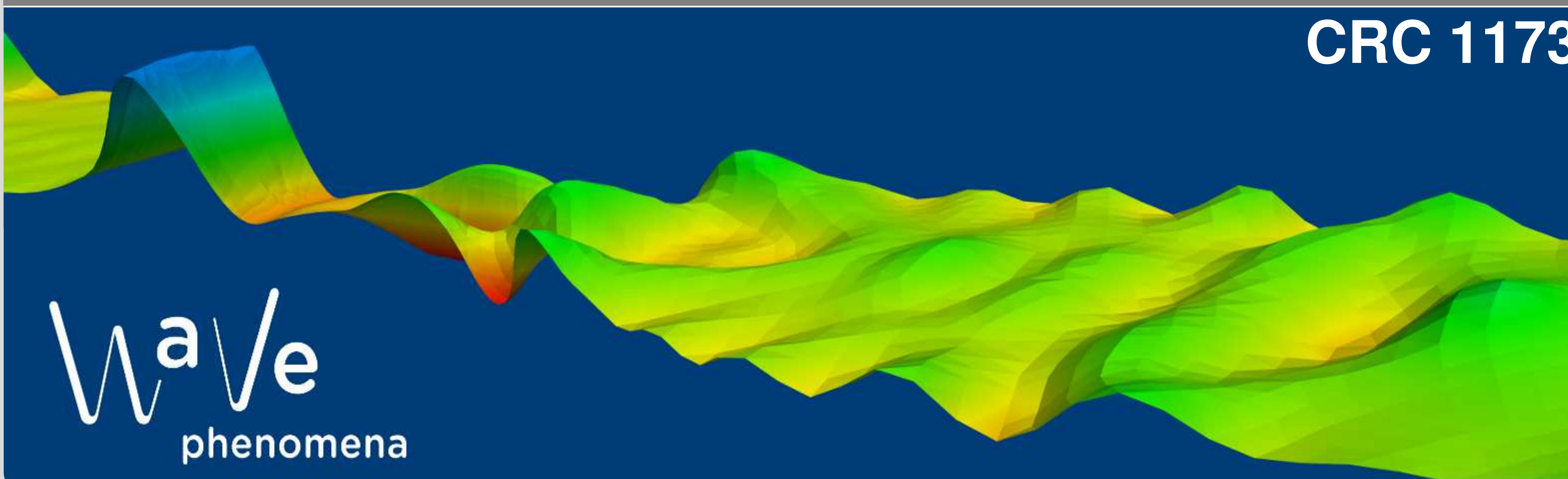
FAKULTÄT FÜR MATHEMATIK – INSTITUT FÜR ANGEWANDTE UND NUMERISCHE MATHEMATIK



Organization of the material

A linear inverse problem in seismic imaging

The elliptic Radon transform

Imaging operators: general results

Imaging operators: concrete setting

Numerical examples

A linear inverse problem in seismic imaging

An inverse problem for the acoustic wave equation

$u(t; \mathbf{x}, \mathbf{x_s})$ acoustic potential in $\mathbf{x} \in \mathbb{R}^d$, $d \in \{2, 3\}$, at time $t \geq 0$

$$\frac{1}{\nu^2}\,\partial_t^2 u - \Delta_{\mathbf{x}} u = \delta(\mathbf{x} - \mathbf{x_s})\delta(t), \qquad u(0, \cdot, \mathbf{x_s}) = \partial_t u(0, \cdot, \mathbf{x_s}) = 0,$$

$\nu = \nu(\mathbf{x})$ speed of sound, $\mathbf{x_s}$ excitation (source) point.

Seismic imaging

Recover ν from the backscattered (reflected) fields

$$u(t; \mathbf{x_r}, \mathbf{x_s}), \quad t \in [0, T], \quad (\mathbf{x_r}, \mathbf{x_s}) \in \mathcal{R} \times \mathcal{S}$$

where

$\mathcal{S}/\mathcal{R}$ sets of source/receiver points, and

T observation period.

Generalized Radon transform

Consider the ansatz

$$\frac{1}{\nu^2(\mathbf{x})} = \frac{1+n(\mathbf{x})}{c^2(\mathbf{x})},$$

$c = c(\mathbf{x})$ smooth and known background velocity.
Determine n from (e.g. SYMES 1998)

$$Fn(T;\mathbf{x_r},\mathbf{x_s}) \approx \int_0^T (T-t)^{d-2}(u-\widetilde{u})(t;\mathbf{x_r},\mathbf{x_s})\mathrm{d}t$$

with the *generalized Radon transform*

$$Fw(T;\mathbf{x_r},\mathbf{x_s}) = \int \frac{w(\mathbf{x})}{c^2(\mathbf{x})} a(\mathbf{x},\mathbf{x_s}) a(\mathbf{x},\mathbf{x_r}) \delta\big(T - \tau(\mathbf{x},\mathbf{x_s}) - \tau(\mathbf{x},\mathbf{x_r})\big)\mathrm{d}\mathbf{x}$$

which integrates w over reflection isochrones: $T = \tau(\cdot,\mathbf{x_s}) + \tau(\cdot,\mathbf{x_r})$.
Travel-time τ and amplitude a can be computed from

$$|\nabla_{\mathbf{x}}\tau| = c^{-1} \quad \text{and} \quad 2\nabla_{\mathbf{x}} a \cdot \nabla_{\mathbf{x}}\tau + a\Delta_{\mathbf{x}}\tau = 0.$$

Historical note: Kirchhoff migration

- Since the 1950's Kirchhoff migration is the standard technique to approximately solve the integral equation.

- BEYLKIN (1984, 1985) gave Kirchhoff migration a mathematical foundation:

$$n_{\text{rec}} = F^{\#} P g \qquad \text{where } g = Fn \text{ are the data (measurements)}$$

P convolution, $F^{\#}$ dual transform (generalized backprojection).

Further,

$$n_{\text{rec}} = F^{\#} P F n = I_{\text{partial}} n + \Psi n,$$

I_{partial} kind of band pass filter, Ψ is smoothing.

Our approach: complementing Kirchhoff

- As we cannot hope to recover n from the data completely we consider imaging operators which differ from the Kirchhoff operator $F^{\#}PF$:

$$\Lambda = KF^{\dagger}\psi F$$

  ψ smooth cutoff,

  $F^{\dagger}$ (weighted) L^2 dual,

  K properly supported pseudo of positive order m.

- We reconstruct Λn from the data $g = Fn$.
- We compute the symbol of Λ to find useful K's.
- Our reconstruction technique based on Λ differs from Kirchhoff migration. In fact, an explicit expression of $F^{\dagger}$ is not required (GRATHWOHL ET AL. 2018)

The elliptic Radon transform

Assumptions

Let $d = 2$ (for the ease of presentation). Further, let

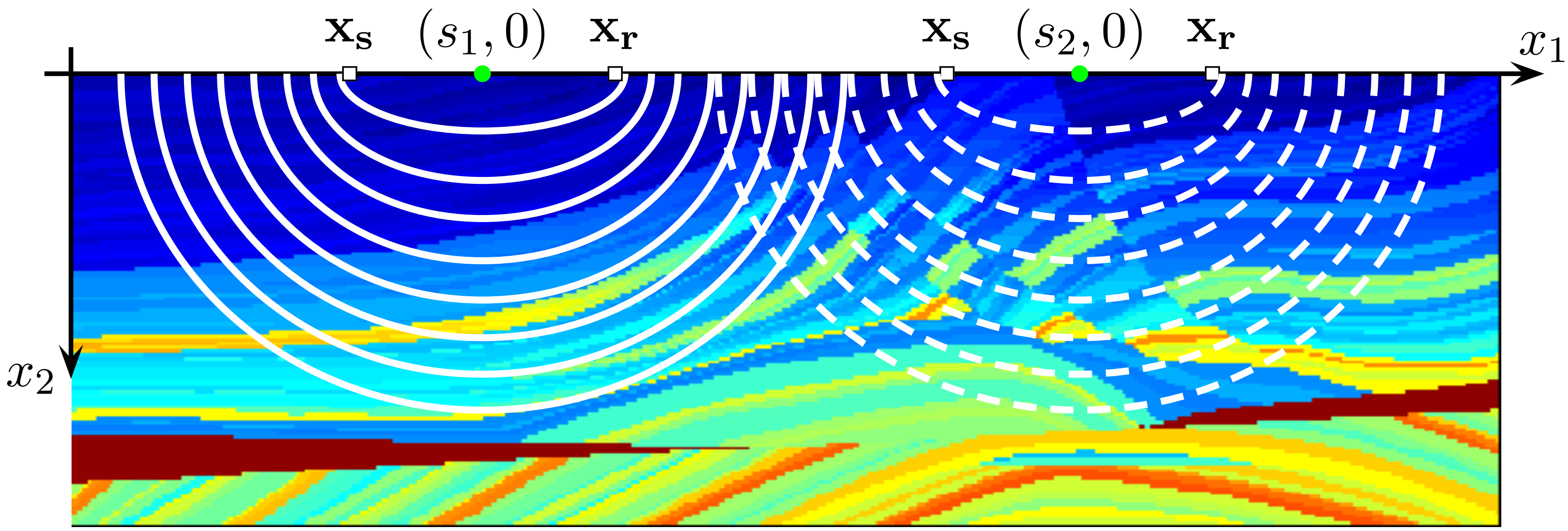
- the background velocity c be constant, say, $c = 1$,
- $n \in L^2(\mathbb{R}^2_+)$ be compactly supported (the positive direction of the x_2-axis points downwards),
- sources and receivers be parameterized by $s \in \mathbb{R}$ via

$$\mathbf{x_s}(s) = (s - \alpha, 0)^\top, \qquad \mathbf{x_r}(s) = (s + \alpha, 0)^\top.$$

with common offset $\alpha \geq 0$ (*common offset data acquisition geometry*).

Thus the reflection isochrones are ellipses with foci $\mathbf{x_s}$ and $\mathbf{x_r}$.

The 2D situation



The elliptic Radon transform

Now the generalized RT becomes

$$Fw(s,t) = \int A(s,\mathbf{x}) w(\mathbf{x}) \delta\big(t - \varphi(s,\mathbf{x})\big) \mathrm{d}\mathbf{x}, \quad t > 2\alpha,$$

with

$$\varphi(s,\mathbf{x}) := |\mathbf{x_s}(s) - \mathbf{x}| + |\mathbf{x_r}(s) - \mathbf{x}|$$

and

$$A(s,\mathbf{x}) = \frac{1}{\sqrt{|\mathbf{x_s}(s) - \mathbf{x}|\,|\mathbf{x_r}(s) - \mathbf{x}|}}.$$

Imaging operators: general results

Basics on pseudo-differential operators 1

Let $\Omega \subset \mathbb{R}^d$ be open and let $P\colon \mathcal{C}^\infty(\Omega) \to \mathcal{C}^\infty(\Omega)$ be a linear differential operator of order m,

$$Pu(\mathbf{x}) = \sum_{|\alpha|\leq m} f_\alpha(\mathbf{x})\partial^\alpha u(\mathbf{x}), \quad f_\alpha \in \mathcal{C}^\infty(\Omega).$$

If $u \in \mathcal{C}_0^\infty(\Omega)$ then

$$Pu(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int \mathrm{e}^{\imath\,\mathbf{x}\cdot\xi} p(\mathbf{x},\xi)\widehat{u}(\xi)\mathrm{d}\xi = \frac{1}{(2\pi)^d} \iint \mathrm{e}^{\imath\,(\mathbf{x}-\mathbf{y})\cdot\xi} p(\mathbf{x},\xi) u(\mathbf{y})\mathrm{d}\mathbf{y}\mathrm{d}\xi$$

where

$$p(\mathbf{x},\xi) = \sum_{|\alpha|\leq m} f_\alpha(\mathbf{x})(\imath\xi)^\alpha$$

is the symbol of P.

Basics on pseudo-differential operators 2

- $p \in \mathcal{C}^\infty(\Omega \times \mathbb{R}^d)$ is a **symbol** of order $m \in \mathbb{R}$ if for any $\alpha, \beta \in \mathbb{N}_0^d$ and any $K \Subset \Omega$ there is a $C = C_{\alpha,\beta,K}$ such that

$$\sup_{\mathbf{x} \in K} |\partial_x^\alpha \partial_\xi^\beta p(\mathbf{x}, \xi)| \le C\,(1 + |\xi|)^{m-|\beta|} \quad \text{for all } \xi \in \mathbb{R}^d.$$

- The set of all symbols of order m on Ω is denoted by $S^m(\Omega)$.
- For $p \in S^m(\Omega)$ we define the operator $\boldsymbol{\Psi}_p$ on $\mathcal{C}_0^\infty(\Omega)$ by

$$\boldsymbol{\Psi}_p u(\mathbf{x}) = \frac{1}{(2\pi)^d} \iint \mathrm{e}^{\imath\,(\mathbf{x}-\mathbf{y})\cdot\xi} p(\mathbf{x}, \xi) u(\mathbf{y}) \mathrm{d}\mathbf{y} \mathrm{d}\xi$$

  and call it a **pseudo-differential operator** of order m.
- The **top order symbol** $\sigma(\boldsymbol{\Psi}_p)$ of $\boldsymbol{\Psi}_p$ is the equivalence class of p in $S^m(\Omega)/S^{m-1}(\Omega)$.

Basics on pseudo-differential operators 3

- Let P be the differential operator with symbol $\sum_{|\alpha|\le m} f_\alpha(\mathbf{x})(\imath\xi)^\alpha$. Then,
$$\sigma(P) = \sum_{|\alpha|=m} f_\alpha(\mathbf{x})(\imath\xi)^\alpha.$$
- The operator $\Lambda^s = (I-\Delta)^{s/2}$ with symbol $(1+|\xi|^2)^{s/2}$ of order $s \in \mathbb{R}$ has top order symbol
$$\varphi(\xi)|\xi|^s$$
for any $\varphi \in \mathcal{C}^\infty$ with $\varphi(\xi) = 1$ for large $|\xi|$ where, in case of $s < 0$, we additionally require that $\varphi = 0$ in a neighborhood of 0.

Basics on pseudo-differential operators 4

Theorem: Let $\mathbf{\Psi}_p$ be a pseudo of order m. Then,

$$\mathbf{\Psi}_p \colon H_0^s(\Omega) \to H_{\mathrm{loc}}^{s-m}(\Omega) \text{ continuously}^1 \text{ for all } s \in \mathbb{R}.$$

Proof: See standard textbooks.

[1] Topology of H^s_{loc}: $u_k \to u$ in H^s_{loc} if $\|\varphi u_k - \varphi u\|_s \to 0$ for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$.

Setting the stage: the imaging operators

We consider

$$KF^\dagger\psi F$$

where

- $\psi\colon S\times(2\alpha,\infty)\to[0,\infty)$ is a smooth compactly supported cutoff fct,
- $F^\dagger$ is the *generalized backprojection* operator

$$F^\dagger u(\mathbf{x}) = \int_S W(s,\mathbf{x})u(s,\varphi(s,\mathbf{x}))\mathrm{d}s$$

  where W is a smooth positive weight, and
- K is a pseudo on $\mathbb{R}^2_+$ of pos. order m with symbol k (properly supp).

Here, S is the bounded open set of parameters $s\in\mathbb{R}$ for the source-receiver pairs.

We have a pseudo

Theorem: Let F, $F^\dagger$, K, ψ, and W be defined as above. Then,

$$KF^\dagger \psi F$$

is a pseudo of order $m - 1$.

Proof:

- GUILLEMIN-STERNBERG 1977: Let R be any hypersurface Radon transform in a d-dim. space and let $R^\dagger$ be its (formal, smoothly weighted) L^2-adjoint. If R satisfies the Bolker assumption then $R^\dagger \psi R$ is a pseudo of order $1 - d$.
- Our transform F on $\mathbb{R}^2_+$ satisfies the Bolker assumption (KRISHNAN ET AL. 2012).
- Hence, $F^\dagger \psi F$ is a pseudo of order -1.
- K is properly supported and has order m.

The symbol

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Theorem: The top order symbol of $KF^\dagger\psi F$ is

$$\sigma(\mathbf{x},\xi) = \frac{2\pi\, k(\mathbf{x},\xi)\psi(s,\varphi(s,\mathbf{x}))W(s,\mathbf{x})A(s,\mathbf{x})}{|\omega| B(s,\mathbf{x})}$$

where

$$B(s,\mathbf{x}) = \left|\det\begin{pmatrix}\nabla_x\varphi(s,\mathbf{x})\\ \frac{\partial}{\partial s}\nabla_x\varphi(s,\mathbf{x})\end{pmatrix}\right|.$$

The symbol is evaluated at $(\mathbf{x},\xi)$ where $s\in\mathbb{R}$ and $\omega\in\mathbb{R}$ are defined by

$$\xi = \omega\nabla_{\mathbf{x}}\varphi(s,\mathbf{x}).$$

$$\varphi(s,\mathbf{x}) = |\mathbf{x_s}(s)-\mathbf{x}| + |\mathbf{x_r}(s)-\mathbf{x}|, \qquad A(s,\mathbf{x}) = \frac{1}{\sqrt{|\mathbf{x_s}(s)-\mathbf{x}|\,|\mathbf{x_r}(s)-\mathbf{x}|}}$$

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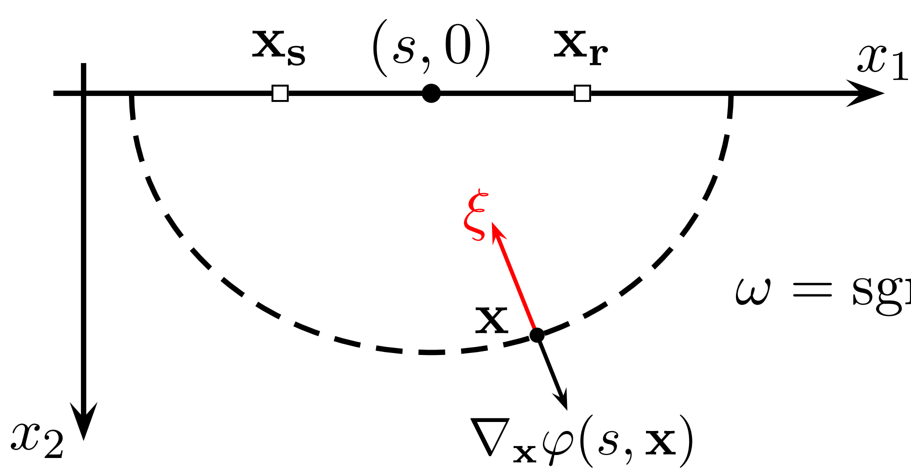

$$\omega = \mathrm{sgn}(\xi_2)\,\frac{|\xi|}{|\nabla_{\mathbf{x}}\varphi(s,\mathbf{x})|}$$

Idea of the proof and related work

- Follow the general calculation of QUINTO 1980:
  He expressed the symbol of generalized Radon transforms in terms of defining measures.

Microlocal properties of F and $F^*\psi F$ in various geometric settings have been studied by several authors, for instance,

BEYLKIN 1985, RAKESH 1988, NOLAN/SYMES 1997,
TEN KRODE ET AL. 1998, STOLK 2000, DE HOOP ET AL. 2009,
QUINTO ET AL. 2011, KRISHNAN ET AL. 2012, FELEA ET AL. 2016,
... etc.

Imaging operators: concrete setting

The symbol: a closer look

Set

$$\Lambda := \Delta F^* \psi F,$$

that is, Λ has order 1 (cf. Bleistein 1987).

Corollary: Let $\alpha = 0$. Then,

$$\sigma(\mathbf{x}, \xi) = -\pi \, \frac{|\xi|}{x_2} \, \psi\Big(x_1 - \tfrac{\xi_1}{\xi_2} \, x_2, 2x_2 \tfrac{|\xi|}{|\xi_2|} \Big).$$

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$$C(\mathbf{x}) := \Big\{ \xi \in \mathbb{R}^2 : \xi_2 \neq 0,\ \psi\Big(x_1 - \tfrac{\xi_1}{\xi_2} x_2, 2x_2 \tfrac{|\xi|}{|\xi_2|} \Big) > 0 \Big\}.$$

Corollary: Let $\alpha = 0$ and $\xi \in C(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2_+$. Then, for $u \in \mathcal{D}'_0(\mathbb{R}^2_+)$

$$(\mathbf{x}, \xi) \in \mathrm{WF}^s(u) \iff (\mathbf{x}, \xi) \in \mathrm{WF}^{s-1}(\Lambda u).$$

$(\mathbf{x}, \xi) \in \mathrm{WF}^s(u) \ :\Longleftrightarrow\ u$ fails to be in H^s about $\mathbf{x}$ in direction ξ

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Corollary: Let $\alpha = 0$ and $\xi \in C(\mathbf{x})$ for $\mathbf{x} \in \mathbb{R}^2_+$. Then, for $u \in \mathcal{D}_0'(\mathbb{R}^2_+)$

$$(\mathbf{x}, \xi) \in \mathrm{WF}^s(u) \iff (\mathbf{x}, \xi) \in \mathrm{WF}^{s-1}(\Lambda u).$$

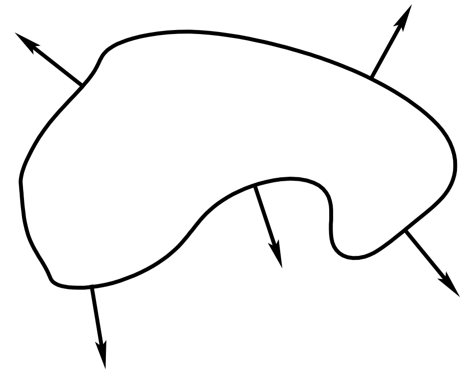
Remark: Similar results hold for $\alpha > 0$.

H^s-Wavefront set: an example

Let $\Omega \subset \mathbb{R}^d$ be open with a $\mathcal{C}^\infty$-boundary. Then

$$\mathrm{WF}^s(\chi_\Omega) = \{(\mathbf{x}, \xi) : \mathbf{x} \in \mathrm{bd}(\Omega),\ \xi \in \mathbb{R}^d \setminus \{\mathbf{0}\},\ \xi \perp \mathrm{bd}(\Omega) \text{ at } \mathbf{x}\},\ s \geq 1/2.$$

For $s < 1/2$ we have $\mathrm{WF}^s(\chi_\Omega) = \emptyset$.



The symbol for $\alpha > 0$

$$\sigma(\mathbf{x},\xi) \approx -\pi \frac{|\xi|}{x_2}\,\psi\Big(x_1 - \tfrac{\xi_1}{\xi_2}\,x_2,\, 2x_2\tfrac{|\xi|}{|\xi_2|}\Big) \quad \text{for } x_2 \gg \alpha$$

$$\sigma(\mathbf{x},\xi) \approx -\frac{\pi}{2}\,\frac{|\xi|^4}{|\xi_2|\,|\xi_1|^2}\,\frac{1}{\alpha}\,\psi\Big(x_1 - \alpha,\, 2\alpha + x_2\,\tfrac{|\xi|^2}{2|\xi_1\xi_2|}\Big) \quad \text{for } \alpha \gg x_2$$

Modified imaging operator

$$\Lambda_{\text{mod}} := \Delta(M + \alpha I)F_2^*\psi F_2$$

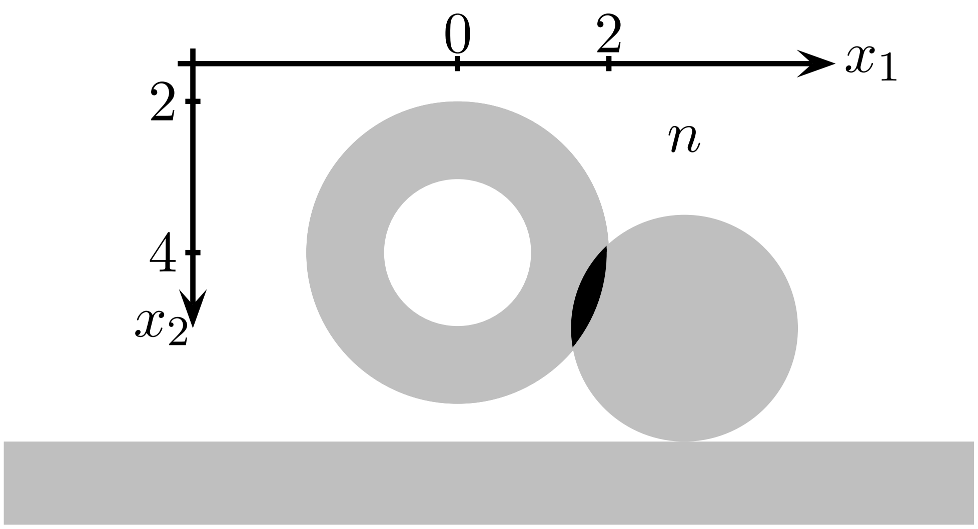
where M is multiplication with x_2 .

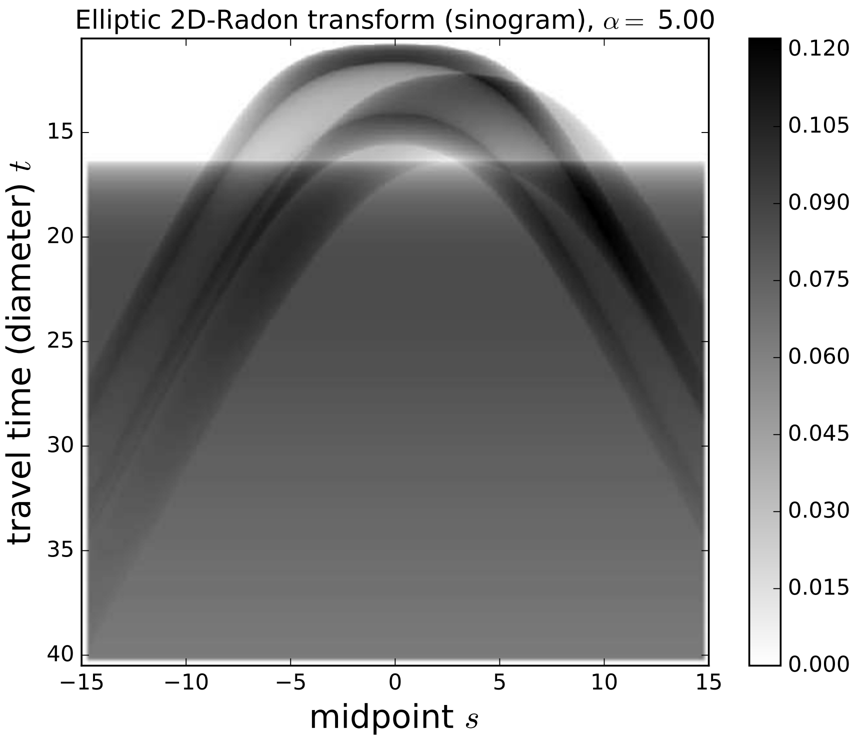
The top order symbol of Λ_{mod} is $(x_2 + \alpha)\sigma(\mathbf{x}, \xi)$.

- The symbol of Λ_{mod} compensates the factor $1/x_2$ for $x_2 \gg \alpha$ and $1/\alpha$ for $\alpha \gg x_2$.
- Thus, jumps in n with the same height but at different depths will be reconstructed with the same intensities relatively independent of α.

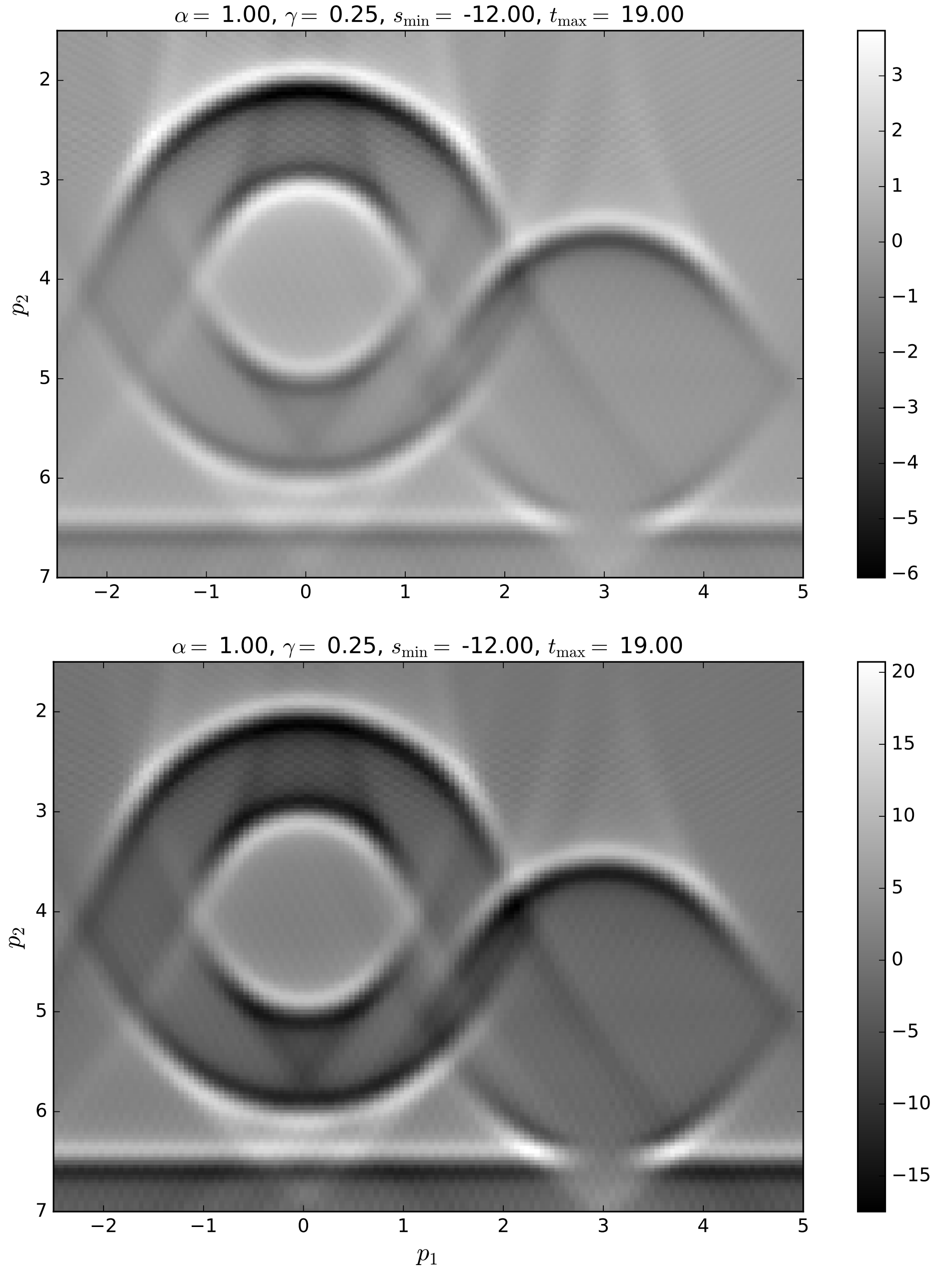
Numerical examples

The phantom n and its transform $\psi F n$

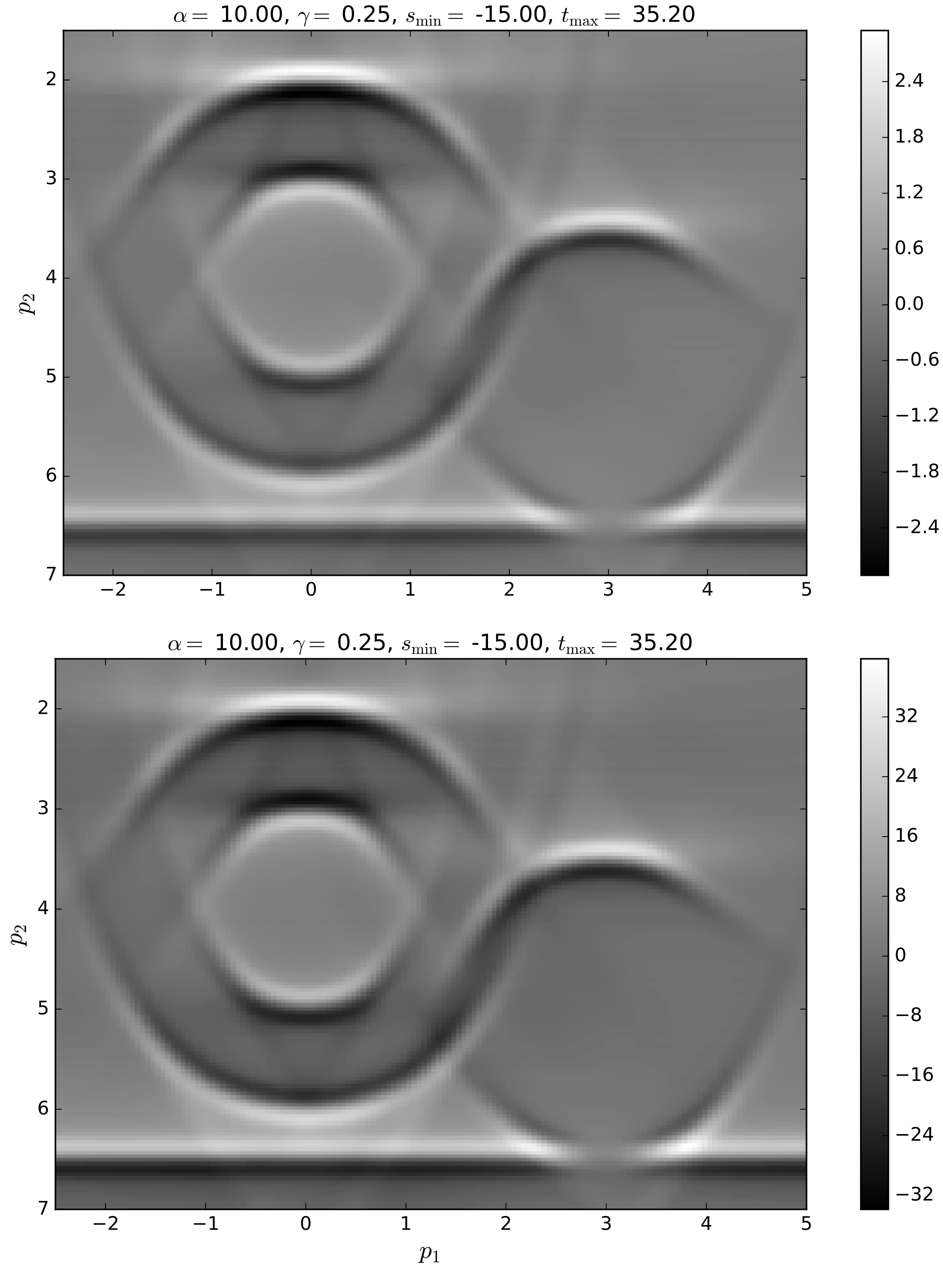




Λ vs. Λ_{mod} for $\alpha = 1$

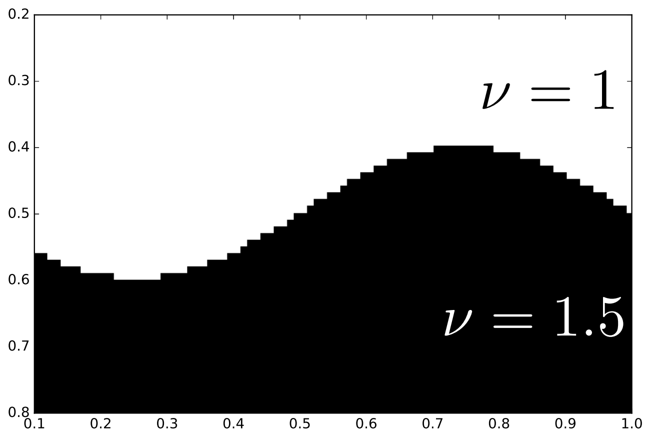


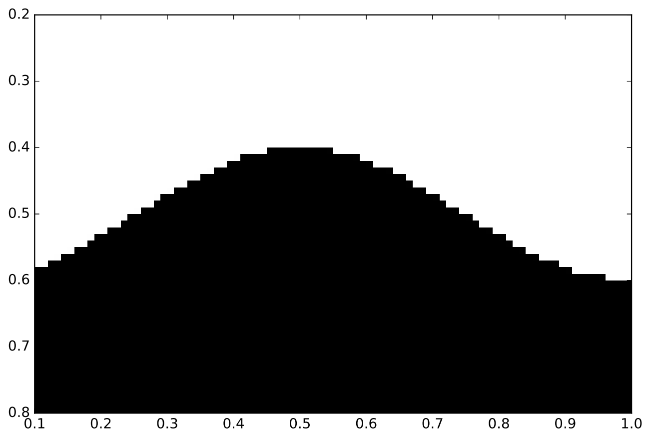
Λ vs. Λ_{mod} for $\alpha = 10$



Data from the wave equation

$$Fw(T;\mathbf{x_r},\mathbf{x_s}) = \int_0^T (u - \widetilde{u})(t;\mathbf{x_r},\mathbf{x_s})\mathrm{d}t$$





- $[0.1, 1] \times [0.1, 0.8]$ with absorbing bc using PML. Step size 0.01.
- 17 source/receiver pairs, $\alpha = 0.05$, positioned at $(s\pm\alpha, 0.1)$, $s \in \{0.15 + 0.05i : i = 0, \ldots, 16\}$, to record u at the receivers.
- Temporal source signal: scaled Gaussian.
- $\widetilde{u}$ was computed with constant sound speed $c = 1$.

Wavefields

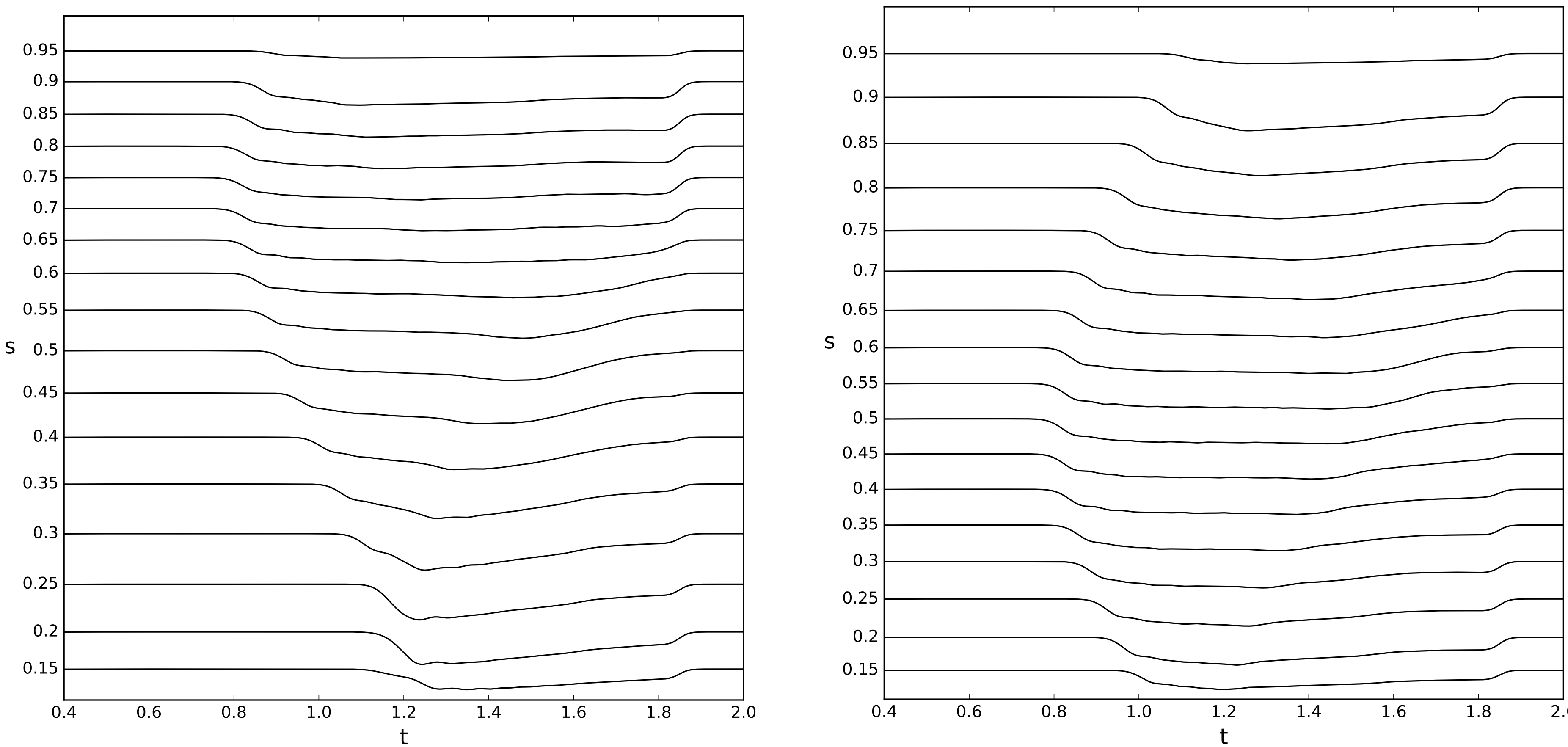
Sine profile

Cosine profile

PySIT –Seismic Imaging Toolbox for Python

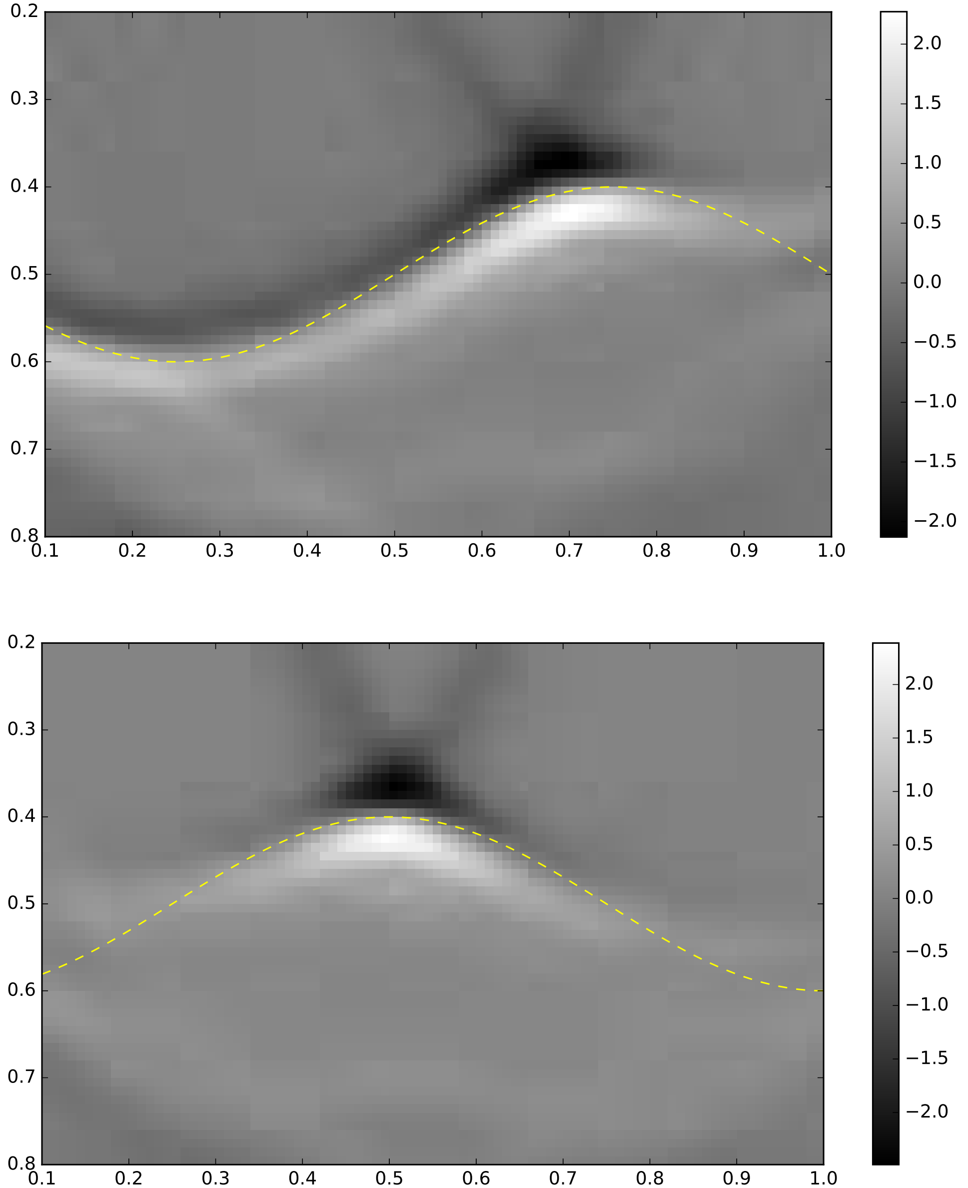
by L. Demanet & R. Hewitt

Preprocessed seismograms

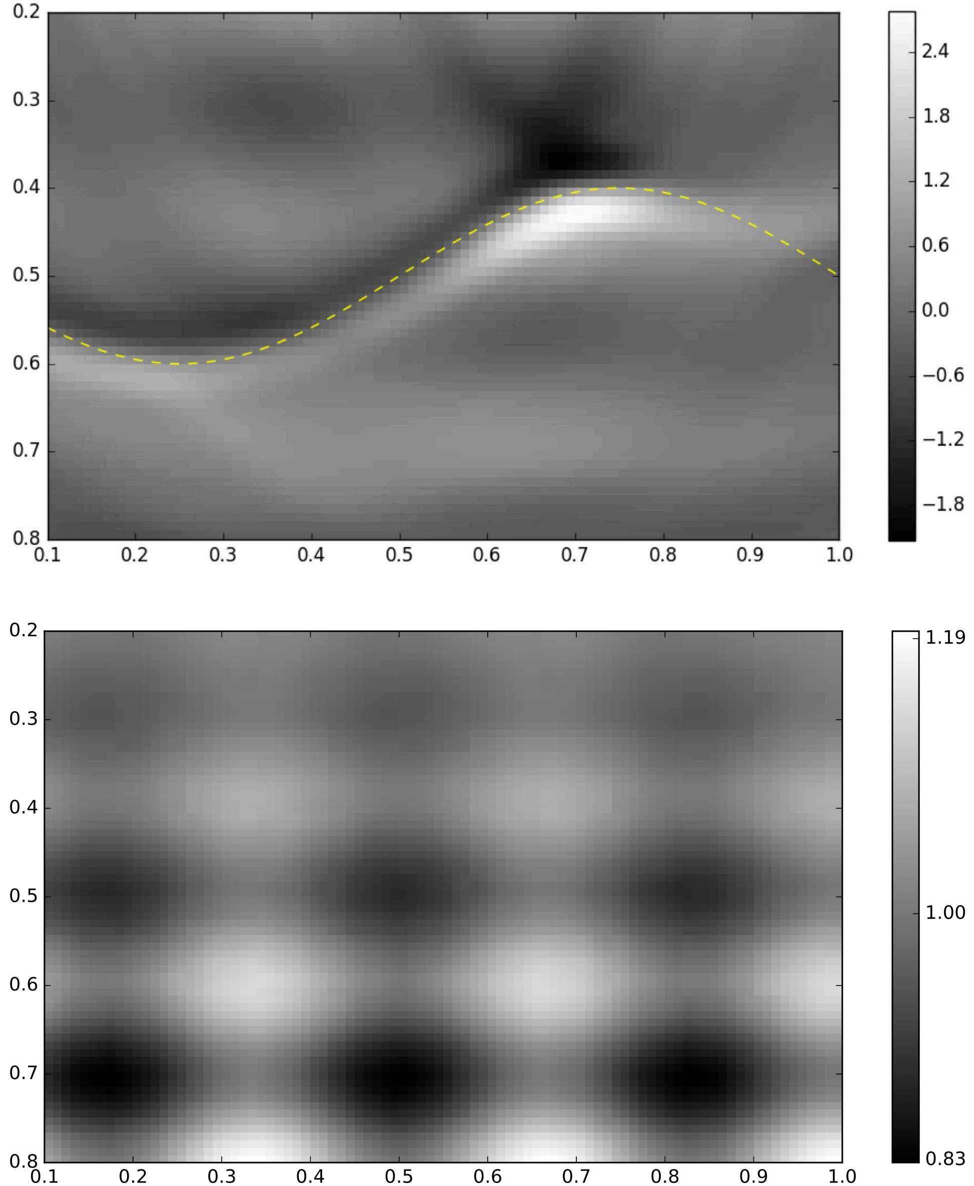


$$y(s,t) = \int_0^T (u - \widetilde{u})(t; \mathbf{x_r}(s), \mathbf{x_s}(s))\mathrm{d}t$$

Reconstructed images



Wrong background velocity



Things to remember

- We introduced a class of imaging operators for the elliptic Radon transform for enhancing singularities.
- These operators are pseudo-differential operators and we computed their symbols explicitly.
- Thus, we constructed operators which reconstruct relatively independently of depth and offset.

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Next challenges

- Non-constant background velocity: symbol, inversion scheme, field data.

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- Non-constant background velocity: symbol, inversion scheme, field data.

Thank you for your attention!

Approximate inverse[2]

Instead of $\Lambda n(\mathbf{p})$ we try to compute

$$\Lambda_\gamma n(\mathbf{p}) := \langle \Lambda n, e_{\mathbf{p},\gamma}\rangle_{L^2(\mathbb{R}^2)} = \Lambda n \star e_{\mathbf{0},\gamma}(\mathbf{p})$$

where $e_{\mathbf{p},\gamma}$, $\gamma > 0$, is a mollifier:

$$\operatorname{supp} e_{\mathbf{p},\gamma} = \overline{B_\gamma(\mathbf{p})}, \qquad \int e_{\mathbf{p},\gamma}(\mathbf{x})\mathrm{d}\mathbf{x} = 1, \qquad e_{\mathbf{p},\gamma} \xrightarrow{\gamma\to 0} \delta(\cdot - \mathbf{p}).$$

We use

$$e_{\mathbf{p},\gamma,k}(\mathbf{x}) = C_{k,\gamma} \begin{cases} (\gamma^2 - \Theta^2)^k & : \ \Theta < \gamma, \\ 0 & : \ \Theta \geq \gamma, \end{cases} \qquad \Theta = |\mathbf{x} - \mathbf{p}|,$$

with $k > 0$ and

$$C_{k,\gamma} = \frac{k+1}{\pi\,\gamma^{2(k+1)}}.$$

---

[2] Louis 1996

Reconstruction kernel

Lemma: For $k \geq 3$ we have that

$$\Lambda_\gamma n(\mathbf{p}) = \langle \Phi F n, \psi_{\mathbf{p},\gamma,k} \rangle_{L^2(\mathbb{R}\times]2\alpha,\infty[)}$$

with the *reconstruction kernel*

$$\psi_{\mathbf{p},\gamma,k}(s,t) = 4k\, C_{k,\gamma} \Big((k-1)\, F\big(|\cdot - \mathbf{p}|^2\, \widetilde{e}_{\mathbf{p},\gamma,k-2} \big)(s,t) - F\widetilde{e}_{\mathbf{p},\gamma,k-1}(s,t) \Big)$$

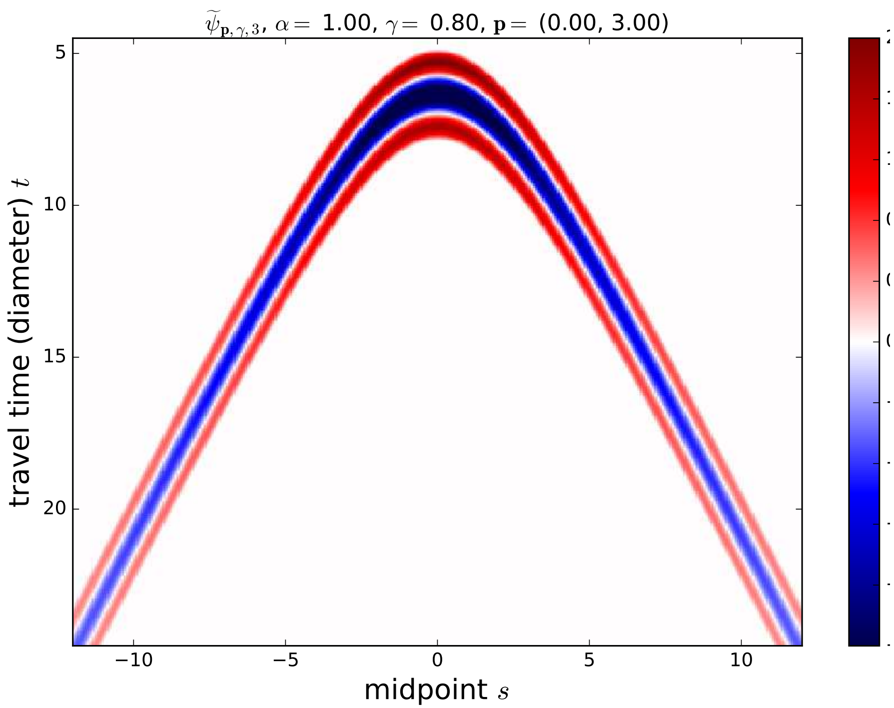
with $\widetilde{e}_{\mathbf{p},\gamma,k} = e_{\mathbf{p},\gamma,k} / C_{k,\gamma}$.

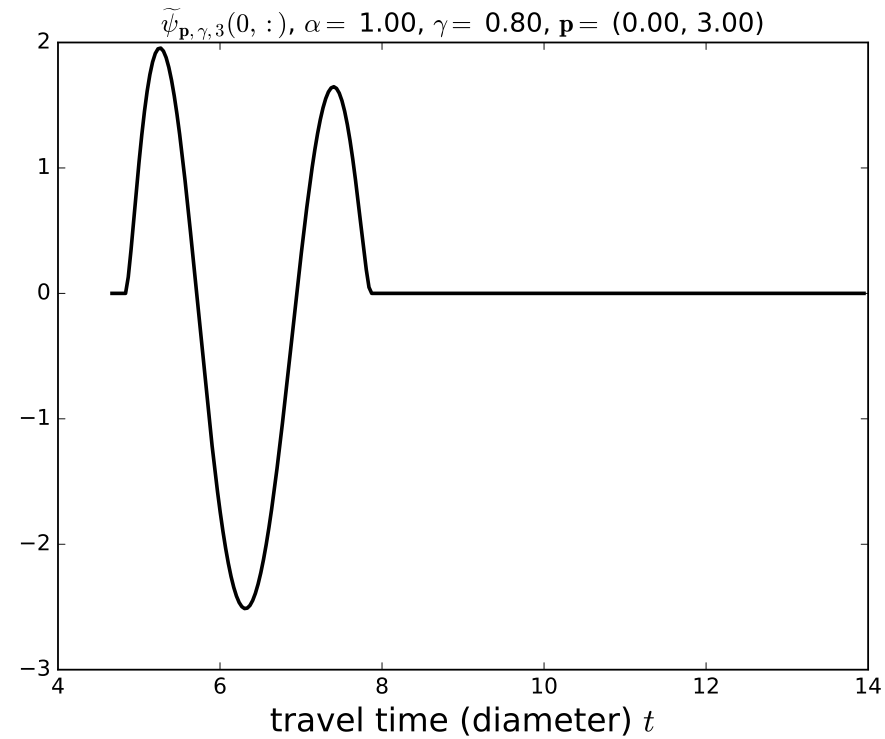
Proof: By duality, $\Lambda_\gamma n(\mathbf{p}) = \langle \Delta F^* \Phi F n, e_{\mathbf{p},\gamma,k} \rangle = \langle \Phi F n, \psi_{\mathbf{p},\gamma,k} \rangle$ with

$$\psi_{\mathbf{p},\gamma,k} = F \Delta e_{\mathbf{p},\gamma,k} = C_{k,\gamma} F \Delta \widetilde{e}_{\mathbf{p},\gamma,k}$$

and $\Delta \widetilde{e}_{\mathbf{p},\gamma,k} = 4k(k-1)\, |\cdot - \mathbf{p}|^2\, \widetilde{e}_{\mathbf{p},\gamma,k-2} - 4k\, \widetilde{e}_{\mathbf{p},\gamma,k-1}$ yields the result. ✓

The kernel for Λ





Discretization

We compute

$$\Lambda_\gamma n(\mathbf{p}) = \langle \Phi F n, \psi_{\mathbf{p},\gamma,3} \rangle_{L^2(\mathbb{R}\times]2\alpha,\infty[)}$$

from the discrete data

$$g(i,j) = \Phi(s_i,t_j) F n(s_i,t_j), \quad i = 1,\ldots,N_s,\ j = 1,\ldots,N_t,$$

where

$$\{s_i\} \subset [-s_{\max}, s_{\max}] \quad \text{and} \quad \{t_j\} \subset [t_{\min}, t_{\max}],\ t_{\min} > 2\alpha,$$

are uniformly distributed with step sizes h_s and h_t, respectively.

$$\Lambda_\gamma n(\mathbf{p}) \approx \widetilde{\Lambda}_\gamma n(\mathbf{p}) := h_s h_t \sum_{i=1}^{N_s} \sum_{t_j \in \mathcal{T}_i(\mathbf{p})} g(i,j)\, \psi_{\mathbf{p},\gamma,3}(s_i,t_j)$$

with $|\mathcal{T}_i(\mathbf{p})| \sim \gamma$.

Computing the kernel

Let χ be the indicator function of $B_r(\mathbf{p})$ which is in the lower half-space. To evaluate

$$F\chi(s,t) = \int A(s,\mathbf{x})\chi(\mathbf{x})\delta\big(t-\varphi(s,\mathbf{x})\big)\mathrm{d}\mathbf{x}, \quad t > 2\alpha,$$

we transform the integral by elliptic coordinates $\mathbf{x}(s,t,\phi) = (x_1,x_2)^\top$,

$$x_1 = s + \frac{t}{2}\cos\phi \quad \text{and} \quad x_2 = \sqrt{\frac{t^2}{4} - \alpha^2}\,\sin\phi.$$

Note: $E(s,t) = \big\{\mathbf{x}(s,t,\phi) : \phi \in [0,2\pi]\big\}$ ellipse wrt $\mathbf{x_s}(s)$, $\mathbf{x_r}(s)$, and t.

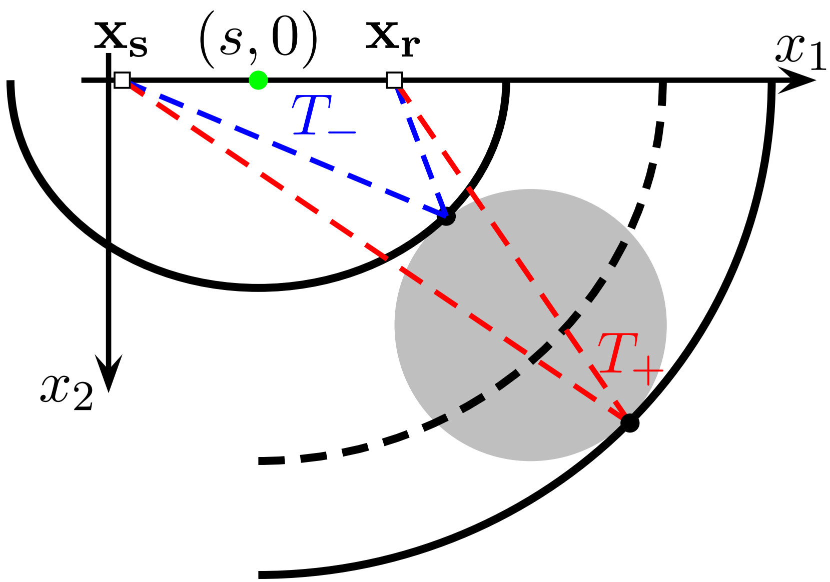
Thus,

$$F\chi(s,t) = \frac{1}{\sqrt{t^2-4\alpha^2}}\int_0^\pi \chi\big(\mathbf{x}(s,t,\phi)\big)\,\mathrm{d}\phi.$$

Computing the kernel (continued)

To evaluate $F\chi(s,t)$ further we provide the following quantities

$$T_{-/+} = T_{-/+}(s,r,\mathbf{p}) = \min / \max \big\{\varphi(s,\mathbf{x}) : \mathbf{x} \in \partial B_r(\mathbf{p})\big\}.$$



$$E(s,t) \cap B_r(\mathbf{p}) \neq \emptyset \iff T_- < t < T_+$$

For $t \in]T_-, T_+[$:

$$E(s,t) \cap B_r(\mathbf{p}) = \big\{\mathbf{x}(s,t,\phi) : \phi \in [\phi_1, \phi_2]\big\}$$

$$F\chi(s,t) = \begin{cases} 0 & : \ t \notin]T_-, T_+[\\ \dfrac{\phi_2 - \phi_1}{\sqrt{t^2 - 4\alpha^2}} & : \ t \in]T_-, T_+[\end{cases}$$

Computing the kernel (continued)

Remaining tasks: Compute $T_{-/+}$, $\phi_{1/2}$.

$$T_{-/+} = \min / \max \left\{ \widetilde{\varphi}(\vartheta) : \vartheta \in [0, 2\pi[\, \right\}$$

where

$$\widetilde{\varphi}(\vartheta) := \varphi\big(s, \mathbf{p} + r(\cos\vartheta, \sin\vartheta)^\top\big).$$

- $\widetilde{\varphi}$ attains exactly one minimum and one maximum in $[0, 2\pi[$.
- As both extrema are clearly separated, we can apply Newton's method to get the two zeros of $\widetilde{\varphi}'$.

Computing the kernel (continued)

- Having $T_\mp$ we solve
$$r^2 = |\mathbf{p} - \mathbf{x}(s,t,\phi)|^2 \quad \text{for} \ \ \phi.$$
For $t \in]T_-, T_+[$, $s \in \mathbb{R}$ we have exactly the two solutions ϕ_1 and ϕ_2.
- We substitute
$$z = \cos\phi, \qquad b = (s - p_1)\, t,$$
$$c = (p_1 - s)^2 + p_2^2 + \frac{t^2}{4} - \alpha^2 - r^2, \qquad d = \sqrt{t^2 - 4\alpha^2}\, p_2,$$
to obtain the equation
$$d\sqrt{1 - z^2} = c + b\, z + \alpha^2\, z^2$$
which has exactly two solutions $-1 \leq z_2 < z_1 \leq 1$.
- By Newton's method again,
$$\phi_i = \arccos z_i, \qquad i = 1, 2.$$

Computing the kernel (continued)

- The kernel $\psi_{\mathbf{p},\gamma,k} = F\Delta e_{\mathbf{p},\gamma,k}$ can be computed just as $F\chi$.
- Indeed, let $k=3$, then

$$\Delta e_{\mathbf{p},\gamma,3}(\mathbf{x}) = C_{3,\gamma}\big(-36\,|\mathbf{x}-\mathbf{p}|^4 + 48\gamma^2\,|\mathbf{x}-\mathbf{p}|^2 - 12\gamma^4\big)\chi_{B_\gamma(\mathbf{p})}(\mathbf{x}).$$

Now F can be applied to each of the components of $\Delta e_{\mathbf{p},\gamma,3}$, e.g.,

$$F\big(|\cdot-\mathbf{p}|^4\chi_{B_\gamma(\mathbf{p})}\big)(s,t) = \begin{cases} 0 & : \ t \notin\,]T_-,T_+[, \\ \dfrac{1}{\sqrt{t^2-4\alpha^2}}\displaystyle\int_{\phi_1}^{\phi_2} |\mathbf{x}(s,t,\phi)-\mathbf{p}|^4\,\mathrm{d}\phi & : \ t \in\,]T_-,T_+[. \end{cases}$$

Here,

$$|\mathbf{x}(s,t,\phi)-\mathbf{p}|^4 = \left(\left(s-p_1+\frac{t}{2}\cos\phi\right)^2 + \left(\sqrt{\frac{t^2}{4}-\alpha^2}\,\sin\phi - p_2\right)^2\right)^2$$

is a trigonometric polynomial which can be integrated analytically.