Allan Cormack + Limited Data Tomography

Todd Quinto

Department of Mathematics

http://math.tufts.edu/faculty/equinto/

Cormack Conference

(Partial support from U.S. NSF, Otto Mønsteds Fond)

August 9, 2019

Tomography

Our Goals: Learn a little bit about Allan Cormack and work on X-ray CT, photoacoustic CT, radar and sonar that followed from our collaborations.

Tomography

Our Goals: Learn a little bit about Allan Cormack and work on X-ray CT, photoacoustic CT, radar and sonar that followed from our collaborations.

Themes:

1. "The hole theorem" [Cormack 1963]–general case: [Helgason 1963, 1965] (see also Gelfand)

Tomography

Our Goals: Learn a little bit about Allan Cormack and work on X-ray CT, photoacoustic CT, radar and sonar that followed from our collaborations.

Themes:

1. "The hole theorem" [Cormack 1963]–general case: [Helgason 1963, 1965] (see also Gelfand)
   - Support theorems [Cormack-Q, Q 1983, Boman-Q, Agranovksy-Q, Krishnan, Kurusa, Gonzalez, Kuchment, Abhishek, Uhlmann, Stefanov, etc.] Allan got me started!
   - Inversion methods for exterior CT [Q 1983, 1988, Perry, Wang, Bates-Lewitt, etc.].

Tomography

Our Goals: Learn a little bit about Allan Cormack and work on X-ray CT, photoacoustic CT, radar and sonar that followed from our collaborations.

Themes:

1. "The hole theorem" [Cormack 1963]–general case: [Helgason 1963, 1965] (see also Gelfand)
   - Support theorems [Cormack-Q, Q 1983, Boman-Q, Agranovksy-Q, Krishnan, Kurusa, Gonzalez, Kuchment, Abhishek, Uhlmann, Stefanov, etc.] Allan got me started!
   - Inversion methods for exterior CT [Q 1983, 1988, Perry, Wang, Bates-Lewitt, etc.].
2. Limited data more generally
   - Visible and invisible features of objects [Q1993, etc.].
   - Added artifacts [Katsevich 1997, Nguyen 2015, Frikel-Q 2013, 2015, Borg-Frikel-Jørgensen-Q 2018, Greenleaf-Uhlmann, Felea-Q, Hahn-Q, Rigaud, Webber, etc.]

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan's big idea: take X-ray pictures from different directions to give "perspective." Pic!

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan's big idea: take X-ray pictures from different directions to give "perspective." Pic!
How to put the pictures together??? **Use math!**

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan's big idea: take X-ray pictures from different directions to give "perspective." Pic!

How to put the pictures together??? **Use math!**

Allan's proof of concept:

- He developed a math algorithm to put the pictures together (2 algorithms + Radon)

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan's big idea: take X-ray pictures from different directions to give "perspective." Pic!

How to put the pictures together??? **Use math!**

Allan's proof of concept:

- He developed a math algorithm to put the pictures together (2 algorithms + Radon)
- He built a simple CT scanner and showed his algorithm worked on real data.

Allan Cormack, Tufts Physicist, Father of CT:

1950s: Allan Cormack, a medical physicist at Groote Schuur Hospital in Cape Town, South Africa, saw doctors taking X-rays pictures of people all the time.

1957-1964, at Tufts: He thought one could do better!

Allan's big idea: take X-ray pictures from different directions to give "perspective." Pic!
How to put the pictures together??? **Use math!**

Allan's proof of concept:

- He developed a math algorithm to put the pictures together (2 algorithms + Radon)
- He built a simple CT scanner and showed his algorithm worked on real data.

Allan won the 1979 Nobel Prize in Medicine! (early call, taught class!)

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.
- He gave me an inversion formula to prove (using spherical harmonics for the spherical transform) that he had already solved *(const.)*

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.
- He gave me an inversion formula to prove (using spherical harmonics for the spherical transform) that he had already solved *(const.)*
- He introduced me to people in the field including our first speaker, Frank Natterer + Alberto Grunbaum, Larry Shepp, . . .

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.
- He gave me an inversion formula to prove (using spherical harmonics for the spherical transform) that he had already solved *(const.)*
- He introduced me to people in the field including our first speaker, Frank Natterer + Alberto Grunbaum, Larry Shepp, . . .
- We worked together on tomography and wrote articles together.

My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.
- He gave me an inversion formula to prove (using spherical harmonics for the spherical transform) that he had already solved *(const.)*
- He introduced me to people in the field including our first speaker, Frank Natterer + Alberto Grunbaum, Larry Shepp, . . .
- We worked together on tomography and wrote articles together.
- He became interested in the pure math side of the subject and we had fun together proving theorems, too. (α and β curves)

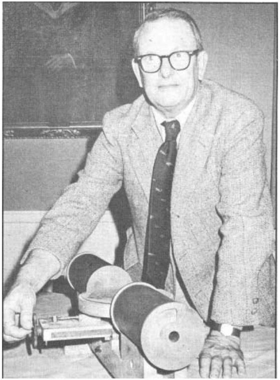
My history with Allan:

- **1975-8:** As an MIT graduate student, I studied Radon transforms (pure–model of X-ray CT), but not tomography (applied).
- **1978:** When I first arrived at Tufts, Allan got me interested in several tomography problems.
- He gave me an inversion formula to prove (using spherical harmonics for the spherical transform) that he had already solved *(const.)*
- He introduced me to people in the field including our first speaker, Frank Natterer + Alberto Grunbaum, Larry Shepp, . . .
- We worked together on tomography and wrote articles together.
- He became interested in the pure math side of the subject and we had fun together proving theorems, too. (α and β curves)

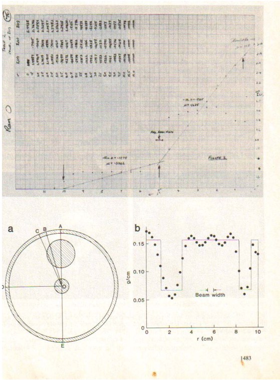
Allan was a modest mensch.

Cormack's CT Scanner

Allan + Scanner

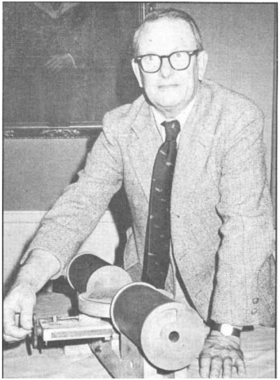


His calculations from the 1960's



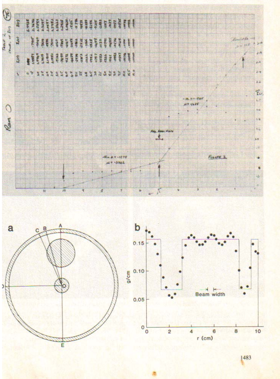
Cormack's CT Scanner

Allan + Scanner



Cost: USD 300

His calculations from the 1960's



Nobel Prize!!

The Mathematical Model of X-ray CT

f a function in the plane representing the density of an object
L a line in the plane over which the photons travel.
Parallel Beam Parameterization:
$L = L(\varphi, p) = \{x \in \mathbb{R}^2 \,|\, x \cdot (\cos(\varphi), \sin(\varphi)) = p\}$, $\varphi \in [-\pi, \pi]$, $p \in \mathbb{R}$.

The Mathematical Model of X-ray CT

f a function in the plane representing the density of an object
L a line in the plane over which the photons travel.
Parallel Beam Parameterization:
$L = L(\varphi, p) = \{x \in \mathbb{R}^2 \big| x \cdot (\cos(\varphi), \sin(\varphi)) = p\}$, $\varphi \in [-\pi, \pi]$, $p \in \mathbb{R}$.

The X-ray (Radon) Transform:

$$\text{Tomographic Data} \sim \mathcal{R}f(\varphi, p) = \int_{x \in L(\varphi, p)} f(x)\,ds$$

–The 'amount' of material on the line the X-rays traverse.

The Mathematical Model of X-ray CT

f a function in the plane representing the density of an object
L a line in the plane over which the photons travel.
Parallel Beam Parameterization:
$L = L(\varphi, p) = \{x \in \mathbb{R}^2 \,|\, x \cdot (\cos(\varphi), \sin(\varphi)) = p\}$, $\varphi \in [-\pi, \pi]$, $p \in \mathbb{R}$.

The X-ray (Radon) Transform:

$$\text{Tomographic Data} \sim \mathcal{R}f(\varphi, p) = \int_{x \in L(\varphi,p)} f(x)\,ds$$

–The 'amount' of material on the line the X-rays traverse.

The goal: Recover a picture of the body (values of $f(x)$), from X-ray CT data over a finite number of lines.

Complete Tomographic Data

Complete Tomographic Data: X-ray data are given over lines throughout the body in directions all around the body.

Limited and Complete Tomographic Data

Complete Tomographic Data: X-ray data are given over lines throughout the body in directions all around the body.

Limited Data Tomography: When some data are missing.

Limited and Complete Tomographic Data

Complete Tomographic Data: X-ray data are given over lines throughout the body in directions all around the body.

Limited Data Tomography: When some data are missing.

Examples:

- *Limited angle X-ray CT:* $Rf(\varphi, p)$ for $a < \varphi < b$ (limited angular range) and all p

Limited and Complete Tomographic Data

Complete Tomographic Data: X-ray data are given over lines throughout the body in directions all around the body.

Limited Data Tomography: When some data are missing.

Examples:

- *Limited angle X-ray CT:* $Rf(\varphi, p)$ for $a < \varphi < b$ (limited angular range) and all p
- *Region of Interest CT:* $Rf(\varphi, p)$ for all φ and $|p| < s$ (for fixed $s > 0$).

Limited and Complete Tomographic Data

Complete Tomographic Data: X-ray data are given over lines throughout the body in directions all around the body.

Limited Data Tomography: When some data are missing.

Examples:

- *Limited angle X-ray CT:* $Rf(\varphi, p)$ for $a < \varphi < b$ (limited angular range) and all p
- *Region of Interest CT:* $Rf(\varphi, p)$ for all φ and $|p| < s$ (for fixed $s > 0$).
- and . . .

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

Can one reconstruct f for $\|x\| \geqslant s$ from this data?

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

Can one reconstruct f for $\|x\| \geqslant s$ from this data?

The Hole Theorem: Yes!

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

Can one reconstruct f for $\|x\| \geqslant s$ from this data?

The Hole Theorem: Yes!

Proofs:

- Allan's first inversion method [Cormack 1963] really solved this problem by inverting Abel-type integral equations for polar Fourier coefficients.

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

Can one reconstruct f for $\|x\| \geqslant s$ from this data?

The Hole Theorem: Yes!

Proofs:

- Allan's first inversion method [Cormack 1963] really solved this problem by inverting Abel-type integral equations for polar Fourier coefficients.
  However, this method did not work numerically so he developed a singular value decomposition (SVD) for the Radon transform with complete data to obtain his reconstructions [Cormack 1964].

The Exterior Problem: When the object is so large (rocket, etc.) that X-rays can't penetrate the inner part of the object *(the "hole").*
Say the object, f, is supported in a circle of radius 1 and $0 < s < 1$.
The data: $Rf(\varphi, p)$ for $\varphi \in [0, 2\pi]$ $|p| \geqslant s$. pic!

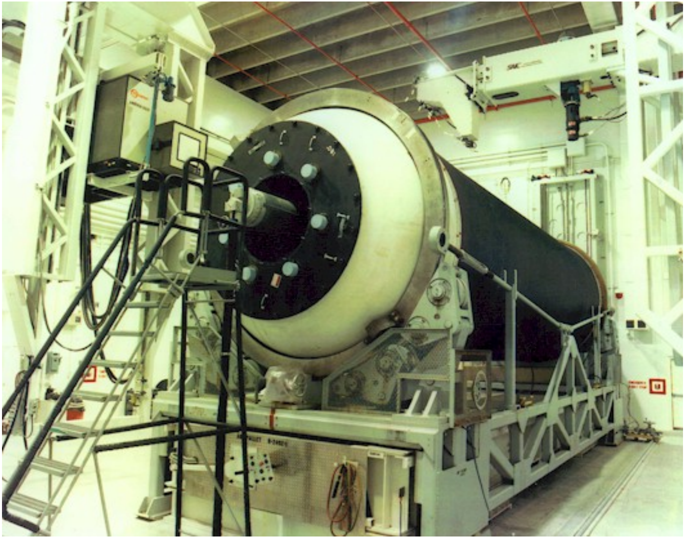
Can one reconstruct f for $\|x\| \geqslant s$ from this data?

The Hole Theorem: Yes!

Proofs:

- Allan's first inversion method [Cormack 1963] really solved this problem by inverting Abel-type integral equations for polar Fourier coefficients.
  However, this method did not work numerically so he developed a singular value decomposition (SVD) for the Radon transform with complete data to obtain his reconstructions [Cormack 1964].
- Independently, Helgason proved the theorem in general at the same time [Helgason 1965] (see also [Gelfand 1966]).

Industrial Exterior CT Scanner [Q 1988, 1998]

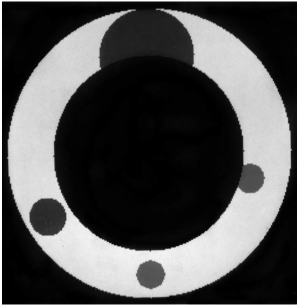


Reconstruction methods for Exterior CT: [Natterer 1980, Bates Lewitt, 1978]

Allan told me about the exterior problem and I developed a SVD [Q 1983] and reconstruction method [Q 1988]. I tested on a phantom related to the one in [Natterer 1980] + industrial data [Q 1998, 2006].

Reconstruction methods for Exterior CT: [Natterer 1980, Bates Lewitt, 1978]

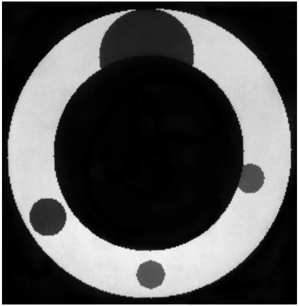
Allan told me about the exterior problem and I developed a SVD [Q 1983] and reconstruction method [Q 1988]. I tested on a phantom related to the one in [Natterer 1980] + industrial data [Q 1998, 2006].



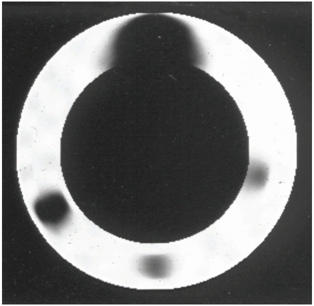
The object

Reconstruction methods for Exterior CT: [Natterer 1980, Bates Lewitt, 1978]

Allan told me about the exterior problem and I developed a SVD [Q 1983] and reconstruction method [Q 1988]. I tested on a phantom related to the one in [Natterer 1980] + industrial data [Q 1998, 2006].



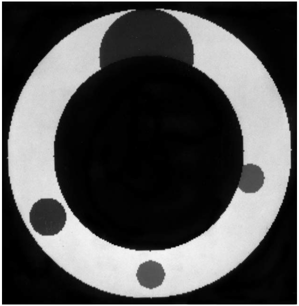
The object

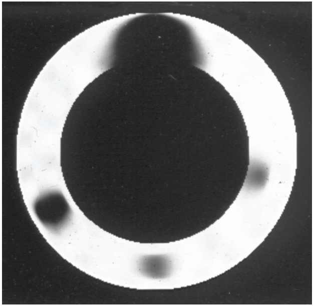


The reconstruction

Reconstruction methods for Exterior CT: [Natterer 1980, Bates Lewitt, 1978]

Allan told me about the exterior problem and I developed a SVD [Q 1983] and reconstruction method [Q 1988]. I tested on a phantom related to the one in [Natterer 1980] + industrial data [Q 1998, 2006].





The object The reconstruction

Is the reconstruction accurate?

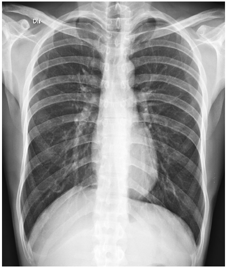
Our plan today

1. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
2. Understand, geometrically, how this depends on the data.

Our plan today

1. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
2. Understand, geometrically, how this depends on the data.

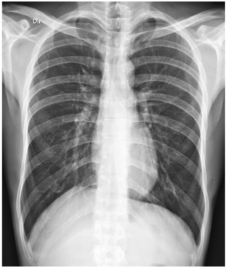
Which parts of the body are sharpest in the X-ray image?



Our plan today

1. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
2. Understand, geometrically, how this depends on the data.

Which parts of the body are sharpest in the X-ray image?
Answer: The edges (boundaries) of the bones and organs!

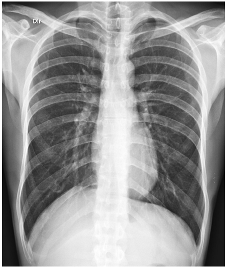


Which X-ray beams show the edges (boundaries)? (pic⟶)

Our plan today

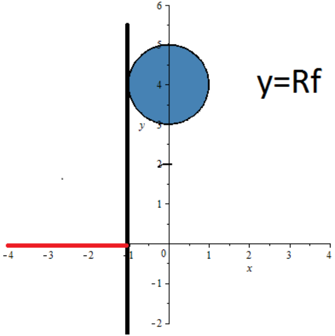
1. Determine what features of the body will be easy to reconstruct from limited CT data, and which will be difficult.
2. Understand, geometrically, how this depends on the data.

Which parts of the body are sharpest in the X-ray image?
Answer: The edges (boundaries) of the bones and organs!

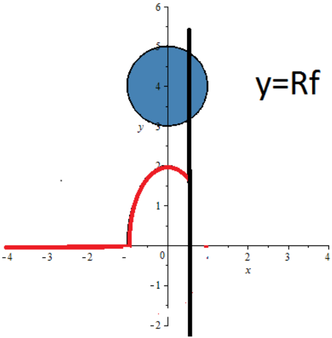


Which X-ray beams show the edges (boundaries)? (pic⟶)
Answer: The beams tangent to the edges (boundaries) of the bones!

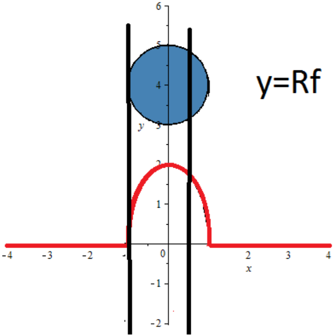
CT Data of cross section of "bone" over vertical lines ($Rf(0,p)$ as a function of p)



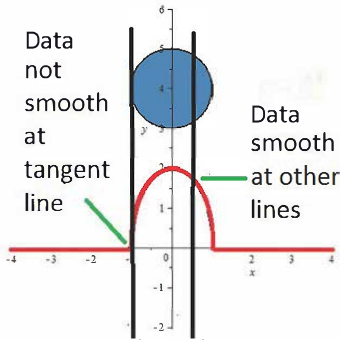
CT Data of cross section of “bone” over vertical lines ($Rf(0, p)$ as a function of p)



CT Data of cross section of “bone” over vertical lines ($Rf(0, p)$ as a function of p)

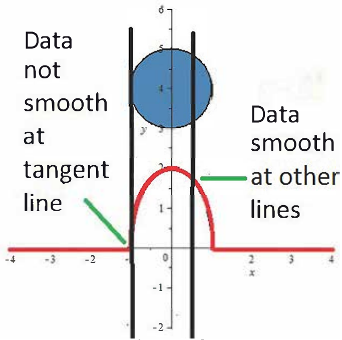


CT Data of cross section of "bone" over vertical lines ($Rf(0, p)$ as a function of p)



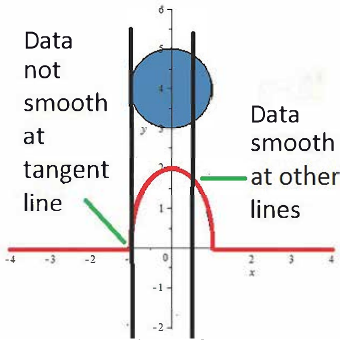
- The CT data has a "corner" (graph not smooth) at a line tangent to the boundary of the disk.

CT Data of cross section of "bone" over vertical lines ($Rf(0, p)$ as a function of p)



- The CT data has a "corner" (graph not smooth) at a line tangent to the boundary of the disk. So the boundary will be easy to see in the data and therefore in the reconstruction.

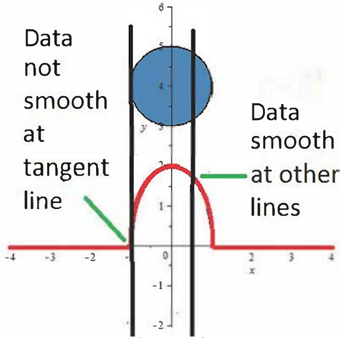
CT Data of cross section of "bone" over vertical lines ($Rf(0,p)$ as a function of p)



- The CT data has a "corner" (graph not smooth) at a line tangent to the boundary of the disk. So the boundary will be easy to see in the data and therefore in the reconstruction.

Moral [Q 1993]: *If a line in the data set is tangent to a boundary of the object, that boundary should be easy to see in the reconstruction.*

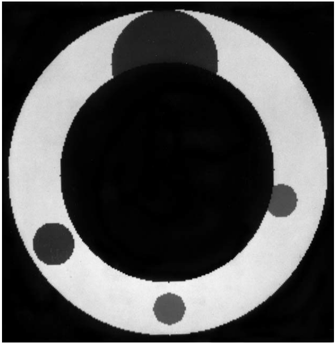
CT Data of cross section of "bone" over vertical lines ($Rf(0, p)$ as a function of p)



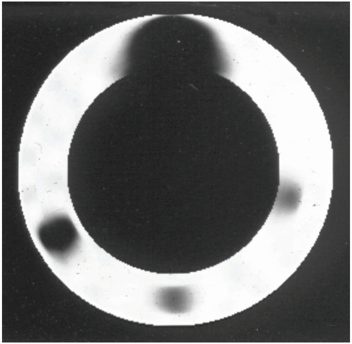
- The CT data has a "corner" (graph not smooth) at a line tangent to the boundary of the disk. So the boundary will be easy to see in the data and therefore in the reconstruction.

Moral [Q 1993]: *If a line in the data set is tangent to a boundary of the object, that boundary should be easy to see in the reconstruction.*
If no line in the data set is tangent to a boundary of the object, that boundary will be hard to see in the reconstruction. *(can be made rigorous!)*

Exterior reconstruction (over line outside the hole) revisited:



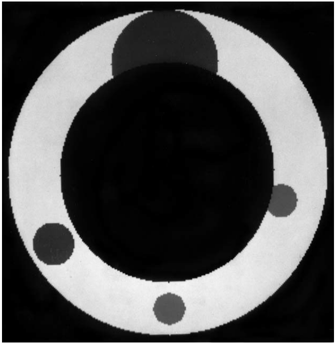
The object



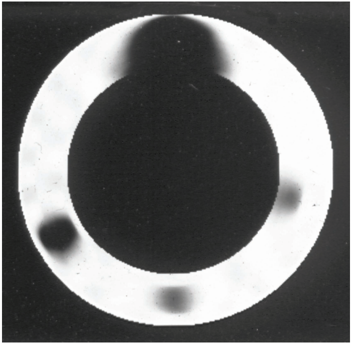
The reconstruction

Which object boundaries are clear in the reconstruction?
Which are not?

Exterior reconstruction (over line outside the hole) revisited:



The object



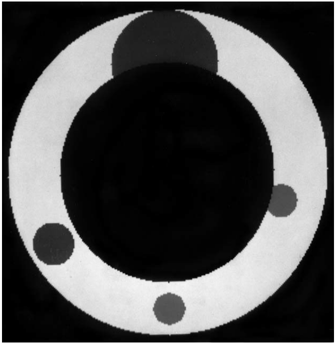
The reconstruction

Which object boundaries are clear in the reconstruction? Which are not?

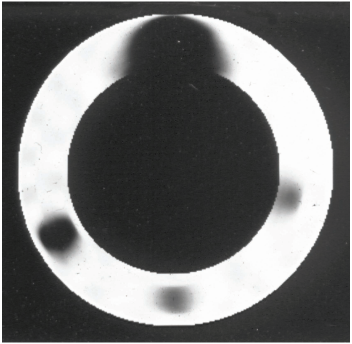
HINT: Which object boundaries are tangent to lines in the exterior data set?

Which object boundaries are tangent to *no line in the data set?*

Exterior reconstruction (over line outside the hole) revisited:



The object



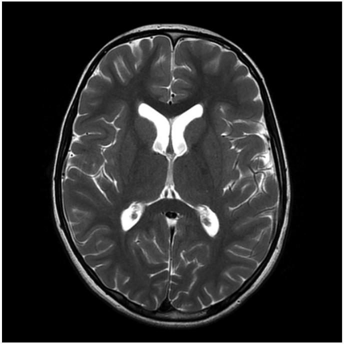
The reconstruction

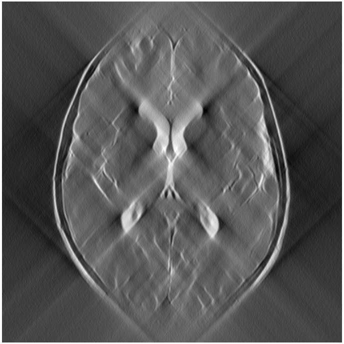
Which object boundaries are clear in the reconstruction? Which are not?

HINT: Which object boundaries are tangent to lines in the exterior data set? *Easy to reconstruct*

Which object boundaries are tangent to *no line in the data set? Hard to reconstruct*

Limited angle data are over lines $L(\varphi, p)$, $a \leqslant \varphi \leqslant b$, $p \in \mathbb{R}$

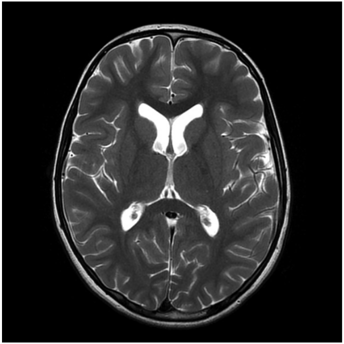


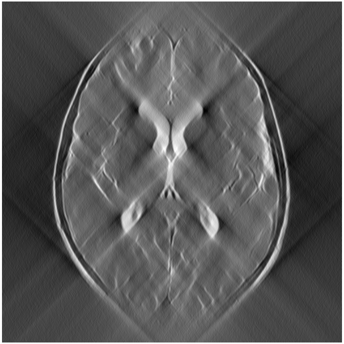


Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

Question: What set of lines would give this limited angle FBP reconstruction?

Limited angle data are over lines $L(\varphi, p)$, $a \leqslant \varphi \leqslant b$, $p \in \mathbb{R}$



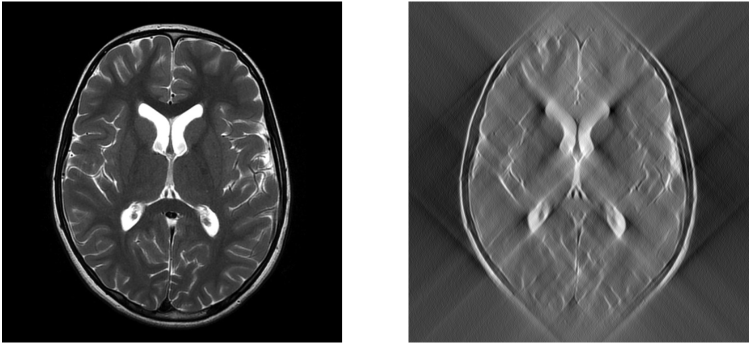


Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

Question: What set of lines would give this limited angle FBP reconstruction?

Hint: Look for visible boundaries and lines tangent to them.

Limited angle data are over lines $L(\varphi, p)$, $a \leqslant \varphi \leqslant b$, $p \in \mathbb{R}$



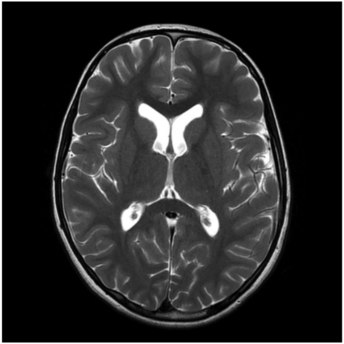
Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

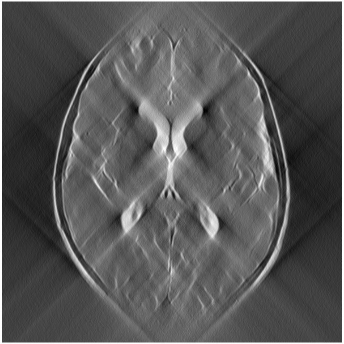
Question: What set of lines would give this limited angle FBP reconstruction?

Hint: Look for visible boundaries and lines tangent to them.

Answer: Vertical-ish lines–lines $L(\varphi, p)$ for $\varphi \in [-45^\circ, 45^\circ]$

Limited angle data are over lines $L(\varphi, p)$, $a \leqslant \varphi \leqslant b$, $p \in \mathbb{R}$





Brain phantom (left) [radiopedia.org], FBP reconstruction [Frikel, Q 2013]

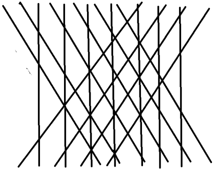
Question: What set of lines would give this limited angle FBP reconstruction?

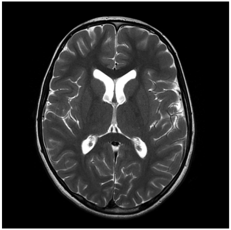
Hint: Look for visible boundaries and lines tangent to them.

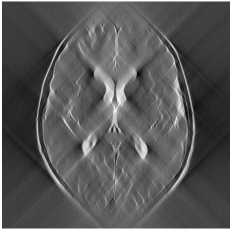
Answer: Vertical-ish lines–lines $L(\varphi, p)$ for $\varphi \in [-45^\circ, 45^\circ]$

Moral: visible boundaries *are* tangent to lines in the data set.

The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.

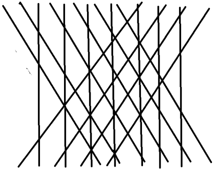


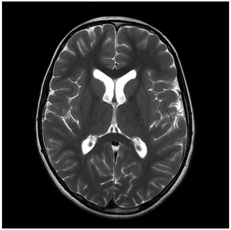


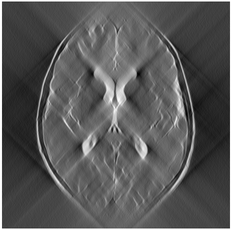


What's up with the streaks???
How do the streaks relate to the data set and object?

The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.



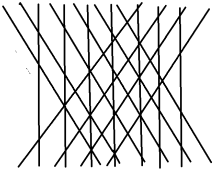
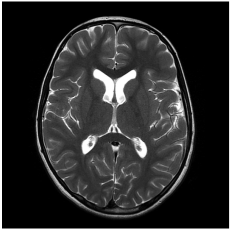
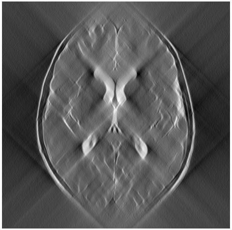




What's up with the streaks???
How do the streaks relate to the data set and object?

They are lines $L(\varphi, p)$. What is their angle φ??

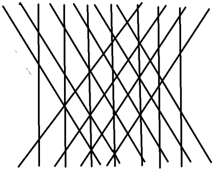
The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.

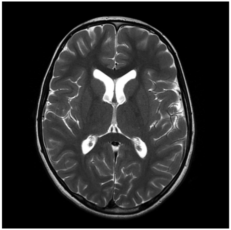




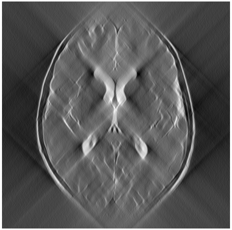
What's up with the streaks???
How do the streaks relate to the data set and object?

They are lines $L(\varphi, p)$. What is their angle φ??
Either -45° or 45°–they are lines at the ends of the data set.

The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.





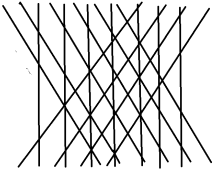
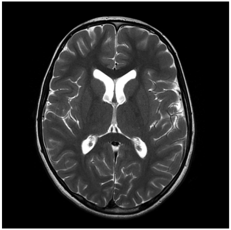
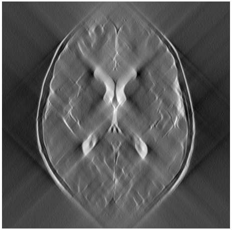


What's up with the streaks???
How do the streaks relate to the data set and object?

They are lines $L(\varphi, p)$. What is their angle φ??
Either -45° or 45°–they are lines at the ends of the data set.

How do the streak lines relate to the object?

The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.

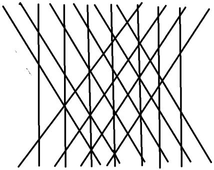




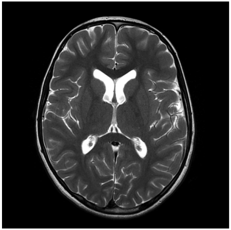
What's up with the streaks???
How do the streaks relate to the data set and object?

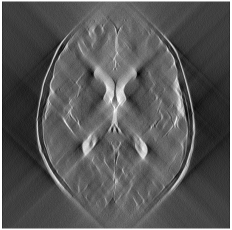
They are lines $L(\varphi, p)$. What is their angle φ??
Either -45° or 45°–they are lines at the ends of the data set.

How do the streak lines relate to the object?
They are tangent to the boundaries of the object.

The data set: $L(\varphi, p)$ for $-45^\circ \leqslant \varphi \leqslant 45^\circ$ and $p \in \mathbb{R}$.







What's up with the streaks???
How do the streaks relate to the data set and object?

They are lines $L(\varphi, p)$. What is their angle φ??
Either -45° or 45°–they are lines at the ends of the data set.

How do the streak lines relate to the object?
They are tangent to the boundaries of the object.

New Moral: Lines at the ends of the data set ($\varphi = -45^\circ$ or $\varphi = 45^\circ$) that are tangent to the object can cause streak artifacts in limited data X-ray CT reconstructions.

Summary so far [Q 1993, Katsevich 1997, Frikel Q 2013, 2015]

- If a boundary of the object *is tangent* to a line in the data set, then it will (should) be *visible* in the reconstruction.

Summary so far [Q 1993, Katsevich 1997, Frikel Q 2013, 2015]

- If a boundary of the object *is tangent* to a line in the data set, then it will (should) be *visible* in the reconstruction.
- If a boundary of the object *is not tangent* to *any* line in the data set, then it should be *invisible* (or at least more difficult to image) in the reconstruction.

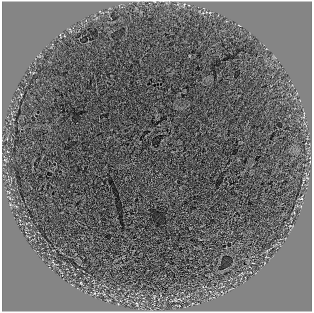
Summary so far [Q 1993, Katsevich 1997, Frikel Q 2013, 2015]

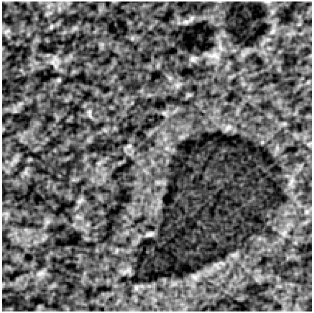
- If a boundary of the object *is tangent* to a line in the data set, then it will (should) be *visible* in the reconstruction.
- If a boundary of the object *is not tangent* to *any* line in the data set, then it should be *invisible* (or at least more difficult to image) in the reconstruction.
- If a boundary of the object *is tangent* to a line *at an end of the data set*, then it can create a streak in the reconstruction along that line.

Summary so far [Q 1993, Katsevich 1997, Frikel Q 2013, 2015]

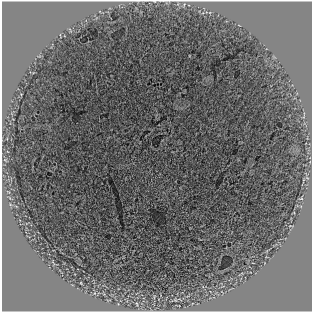
- If a boundary of the object *is tangent* to a line in the data set, then it will (should) be *visible* in the reconstruction.
- If a boundary of the object *is not tangent* to *any* line in the data set, then it should be *invisible* (or at least more difficult to image) in the reconstruction.
- If a boundary of the object *is tangent* to a line *at an end of the data set*, then it can create a streak in the reconstruction along that line.
- These observations are made precise and proven using deep mathematics (Fourier and microlocal analysis).

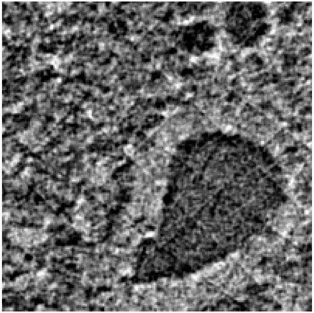
X-ray CT: *In Denmark, I recently saw a crazy limited data synchrotron CT reconstruction [BFJ].*





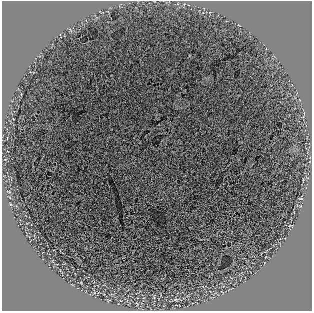
X-ray CT: *In Denmark, I recently saw a crazy limited data synchrotron CT reconstruction [BFJ].*

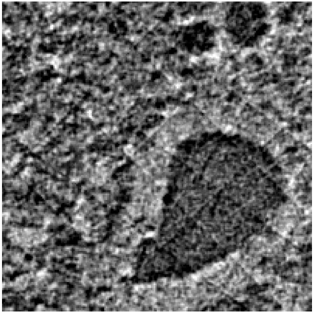




- *It did not fit our theory!* The streaks were not tangent to boundaries of the object!

X-ray CT: *In Denmark, I recently saw a crazy limited data synchrotron CT reconstruction [BFJ].*





- *It did not fit our theory!* The streaks were not tangent to boundaries of the object!
- To explain it, we developed a theory for FBP for *all* limited data X-ray CT problems [Borg, Frikel, Jørgensen, Q, SIIMS 2018]–Jürgen Frikel's minisymposium talk on Monday.

Math Model of Photoacoustic Tomography (PAT)

Circular mean Radon transform in 2-dimensions:

$$\mathcal{M}f(\xi, r) = \frac{1}{2\pi} \int_{S^1} f(\xi + r\mathbf{u})\, \mathrm{d}(\mathbf{u}) \qquad \xi \in \mathbb{R}^2.$$

Math Model of Photoacoustic Tomography (PAT)

Circular mean Radon transform in 2-dimensions:

$$\mathcal{M}f(\xi, r) = \frac{1}{2\pi} \int_{S^1} f(\xi + r\mathbf{u})\, \mathrm{d}(\mathbf{u}) \qquad \xi \in \mathbb{R}^2.$$

2D circular mean Radon transform is the model in sectional PAT imaging setups: pulsed radiation is beamed into the object which then heats up, expands, and releases ultrasonic waves. They are detected by ultrasound detectors around the object [Razansky 2009, Elbau 2012].

Math Model of Photoacoustic Tomography (PAT)

Circular mean Radon transform in 2-dimensions:

$$\mathcal{M}f(\xi, r) = \frac{1}{2\pi}\int_{S^1} f(\xi + r\mathbf{u})\,\mathrm{d}(\mathbf{u}) \qquad \xi \in \mathbb{R}^2.$$

2D circular mean Radon transform is the model in sectional PAT imaging setups: pulsed radiation is beamed into the object which then heats up, expands, and releases ultrasonic waves. They are detected by ultrasound detectors around the object [Razansky 2009, Elbau 2012].

Pure mathematics for spherical transforms: John, Berenstein, Zalcman, Agranovsky, Kuchment, Kunyansky, Q, Lin, Zobin, Ambartsoumian, Nguyen, Krishnan....

Math Model of Photoacoustic Tomography (PAT)

Circular mean Radon transform in 2-dimensions:

$$\mathcal{M}f(\xi, r) = \frac{1}{2\pi} \int_{S^1} f(\xi + r\mathbf{u})\, \mathrm{d}(\mathbf{u}) \qquad \xi \in \mathbb{R}^2.$$

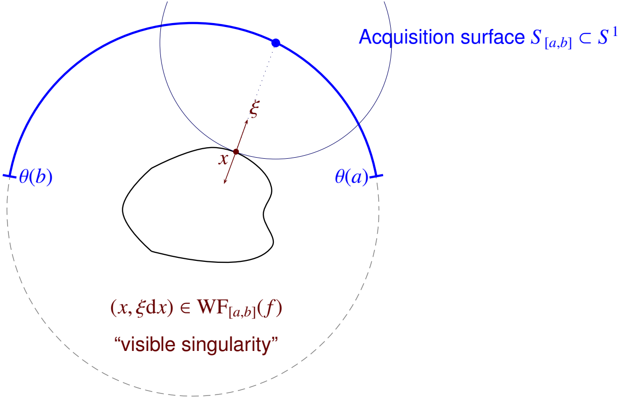
2D circular mean Radon transform is the model in sectional PAT imaging setups: pulsed radiation is beamed into the object which then heats up, expands, and releases ultrasonic waves. They are detected by ultrasound detectors around the object [Razansky 2009, Elbau 2012].

Pure mathematics for spherical transforms: John, Berenstein, Zalcman, Agranovsky, Kuchment, Kunyansky, Q, Lin, Zobin, Ambartsoumian, Nguyen, Krishnan....
Reconstruction methods: Kunyansky, Louis et al., Finch, Patch, Rakesh, Bal, Schotland, Scherzer, Ren, Kocyigit, Ambartsoumian, Krishnan, Q-Rieder-Schuster,....

Visible and Invisible Boundaries and Artifacts

Radon transforms detect singularities normal to the set being integrated over [Guillemin Sternberg]

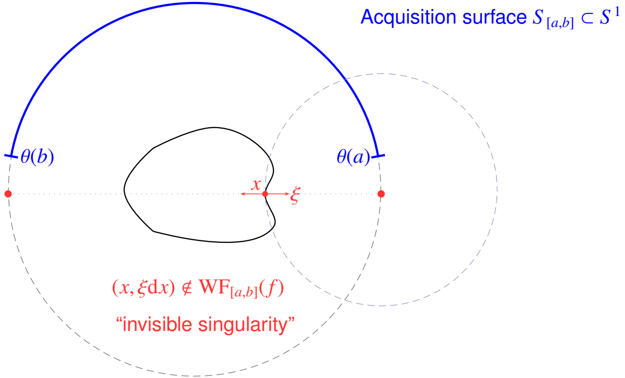
Visible boundaries—tangent to some circle in the data set.



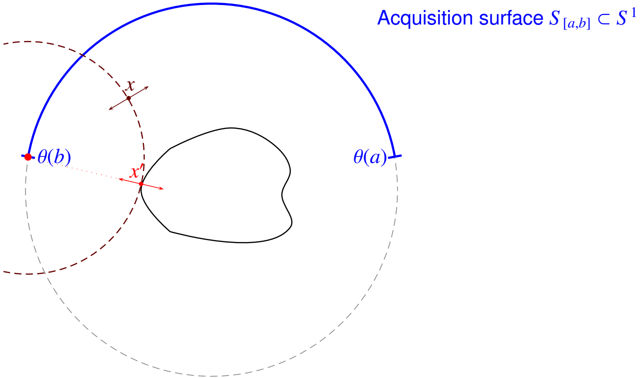
Visible and Invisible Boundaries and Artifacts

Radon transforms detect singularities normal to the set being integrated over [Guillemin Sternberg]

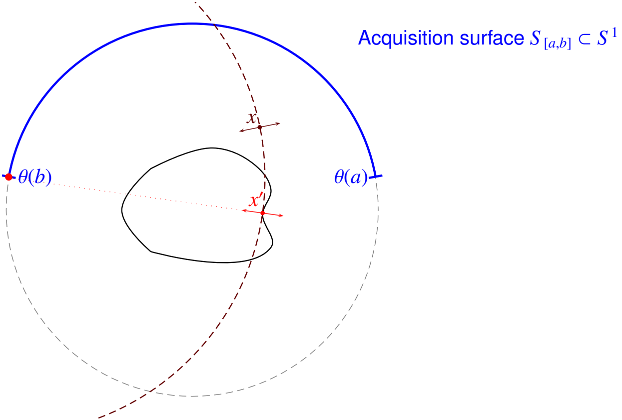
Invisible boundaries–not tangent to any circle in the data set.



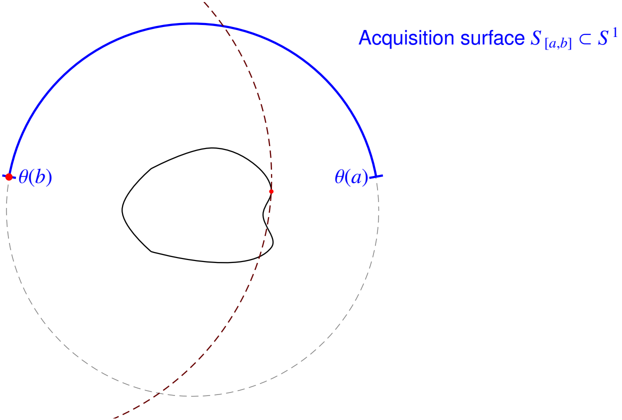
If $\phi = a$ or $\phi = b$ *(bndy of data set)* and if some singularity in WF(f) is normal a circle $C(\phi, r)$, then **artifacts can be added on the entire circle.**



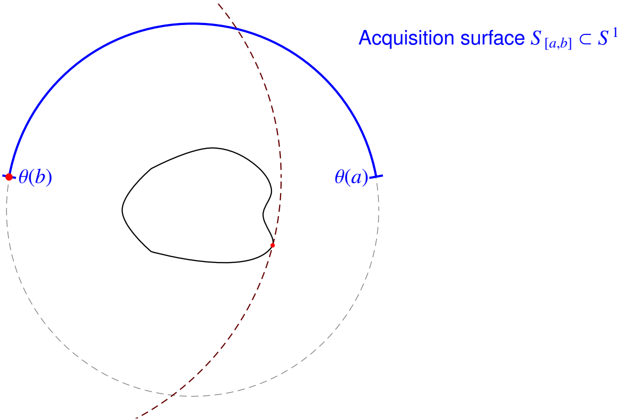
If $\phi = a$ or $\phi = b$ *(bndy of data set)* and if some singularity in $\mathrm{WF}(f)$ is normal a circle $C(\phi, r)$, then **artifacts can be added on the entire circle.**



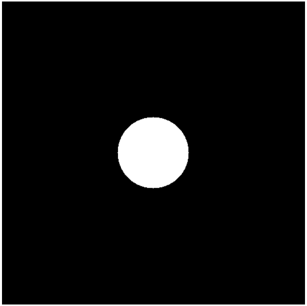
If $\phi = a$ or $\phi = b$ *(bndy of data set)* and if some singularity in WF(f) is normal a circle $C(\phi, r)$, then **artifacts can be added on the entire circle.**



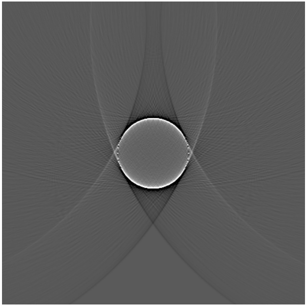
If $\phi = a$ or $\phi = b$ *(bndy of data set)* and if some singularity in WF(f) is normal a circle $C(\phi, r)$, then **artifacts can be added on the entire circle.**



Limited view reconstructions



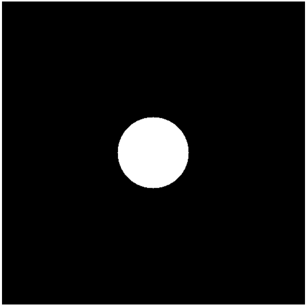
(a) f



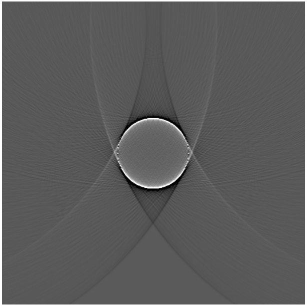
(b) $\Lambda g = \mathcal{M}^* \left(-\frac{d^2}{dr^2} g \right)$

Lambda reconstruction for **range of view** $[25^\circ, 155^\circ]$. Note the added artifacts are along circles at $\theta(25^\circ)$ and $\theta(155^\circ)$.

Limited view reconstructions



(c) f

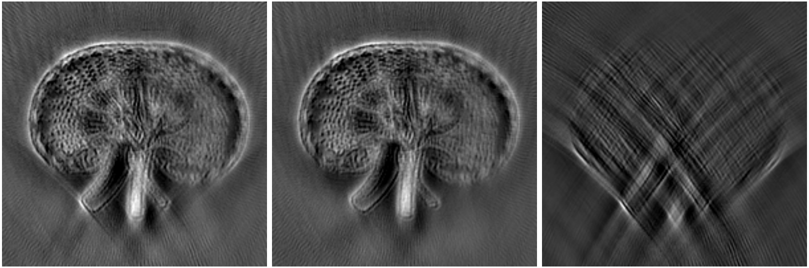


(d) $\Lambda g = \mathcal{M}^* \left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} g \right)$

Lambda reconstruction for **range of view** $[25^\circ, 155^\circ]$. Note the added artifacts are along circles at $\theta(25^\circ)$ and $\theta(155^\circ)$.

A simple artifact reduction method turns $\mathcal{M}^* P \mathcal{M}_{[a,b]}$ into a standard ΨDO.

Real data reconstructions



(e) no artifact reduction (f) with artifact reduction (g) difference

Paper phantom with ink as acoustic absorber[1] .

Reconstruction of singularities for the **range of view** $[-45^\circ, 225^\circ]$ **(all sings. are visible!)**.

[1] Data by courtesy of Prof. Daniel Razansky (Institute of Biological and Medical Imaging, Helmholtz Zentrum München).

A problem in Bistatic Synthetic Aperture Radar (SAR)

In Bistatic SAR, the transmitter and receiver are on separate platforms so they can be more difficult to detect. The goal is to map the surface of the earth that they observe.

Under the Born approximation and constant speed of propagation (and height = 0) the forward map from the object to the data is an FIO with the same microlocal properties as an elliptical Radon transform with foci the transmitter and receiver.

Related microlocal work: Cheney, Nolan, Borden, Felea, Greenleaf, Krishnan, Q, Levinson, Ambartsoumian, Q, Stefanov, Uhlmann, Yazici, S., Moon, Heo,. . .

Bistatic SAR with fixed transmitter

The big idea: Use a fixed transmitter already in the area such as a cell phone or radio tower and have the receiver be on a drone that flies independently around it.

Bistatic SAR with fixed transmitter

The big idea: Use a fixed transmitter already in the area such as a cell phone or radio tower and have the receiver be on a drone that flies independently around it.

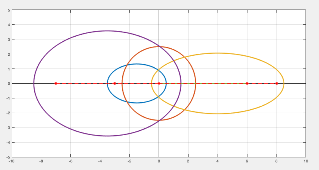
The question: How should the drone fly to get the most complete picture of the scene?

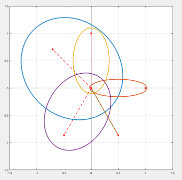
Bistatic SAR with fixed transmitter

The big idea: Use a fixed transmitter already in the area such as a cell phone or radio tower and have the receiver be on a drone that flies independently around it.

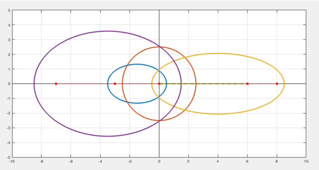
The question: How should the drone fly to get the most complete picture of the scene?
As a first step, analyze which path gives the best reconstructions for the elliptical transform.

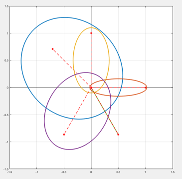
Junior Ally Lee and Sophomore Gloria Kitchens worked with me to evaluate flight paths for receiver.

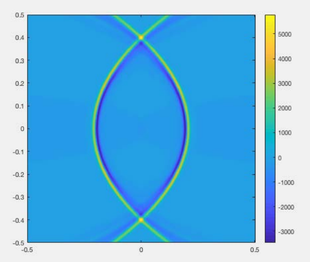


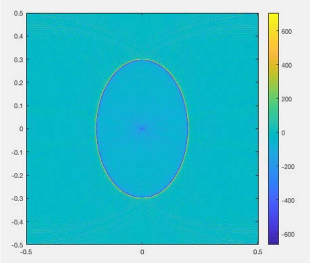


Junior Ally Lee and Sophomore Gloria Kitchens worked with me to evaluate flight paths for receiver.
Here is a reconstruction of an ellipse when the receiver travels on a line (left) and on a circle (right). Which is better?

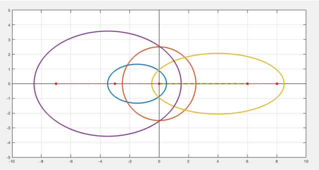


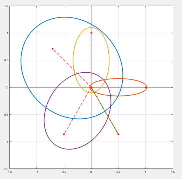


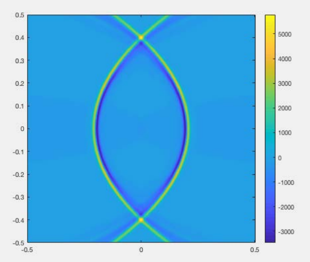


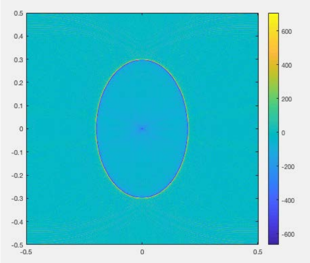


Junior Ally Lee and Sophomore Gloria Kitchens worked with me to evaluate flight paths for receiver.
Here is a reconstruction of an ellipse when the receiver travels on a line (left) and on a circle (right). Which is better? Now, we have a conjecture, so I am working with grad student Jon Warneke to prove it.









Conclusion

- This work is based on microlocal analysis–a precise characterization of singularities and it is valid for a range of tomography problems

Conclusion

- This work is based on microlocal analysis–a precise characterization of singularities and it is valid for a range of tomography problems
- We prove our theory for filtered backprojection type reconstruction algorithms, but the visible, invisible singularities (and artifacts) are seen in other inversion methods

Conclusion

- This work is based on microlocal analysis–a precise characterization of singularities and it is valid for a range of tomography problems
- We prove our theory for filtered backprojection type reconstruction algorithms, but the visible, invisible singularities (and artifacts) are seen in other inversion methods—-without additional assumptions (regularization) on the object such as
    - piecewise constant (total variation)
    - related to a learning data set (machine learning)

Conclusion

- This work is based on microlocal analysis–a precise characterization of singularities and it is valid for a range of tomography problems
- We prove our theory for filtered backprojection type reconstruction algorithms, but the visible, invisible singularities (and artifacts) are seen in other inversion methods—-without additional assumptions (regularization) on the object such as
  - piecewise constant (total variation)
  - related to a learning data set (machine learning)

  Wish: maybe one can join these ideas together to, for example, focus on certain data in the training data set.

Tufts UNIVERSITY

Conclusion

- This work is based on microlocal analysis–a precise characterization of singularities and it is valid for a range of tomography problems

- We prove our theory for filtered backprojection type reconstruction algorithms, but the visible, invisible singularities (and artifacts) are seen in other inversion methods—-without additional assumptions (regularization) on the object such as
  - piecewise constant (total variation)
  - related to a learning data set (machine learning)

  Wish: maybe one can join these ideas together to, for example, focus on certain data in the training data set.

Thanks for coming to the conference!!

For Further Reading I

General references:

- Frank Natterer, *The Mathematics of Computerized Tomography*, Wiley, New York, 1986 (SIAM 2001).
- Frank Natterer, Frank Wuebbling, *Mathematical Methods in Image Reconstruction*, SIAM, 2001.

Introductory

- Peter Kuchment, The Radon transform and medical imaging. CBMS-NSF Regional Conference Series in Applied Mathematics, 85. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2014. xvi+240 pp.

For Further Reading II

- E.T. Quinto, An Introduction to X-ray tomography and Radon Transforms, Proceedings of Symposia in Applied Mathematics, Vol. 63, 2006, pp. 1-24.

Local and Lambda CT

- A. Faridani, E.L. Ritman, and K.T. Smith, *SIAM J. Appl. Math.* **52**(1992), 459–484,
+Finch II: **57**(1997) 1095–1127.
- A. Katsevich, Cone Beam Local Tomography, SIAM J. Appl. Math. 1999, Improved: Inverse Problems 2006.
- A. Louis and P. Maaß, *IEEE Trans. Medical Imaging*, 12(1993), 764-769.

For Further Reading III

Microlocal references:

- *Intro + Microlocal:* Microlocal Analysis in Tomography, joint with Venkateswaran Krishnan, chapter in Handbook of Mathematical Methods in Imaging, 2e, pp. 847-902, Editor Otmar Scherzer, Springer Verlag, New York, 2015 www.springer.com/978-1-4939-0789-2

- Petersen, Bent E., Introduction to the Fourier transform & pseudodifferential operators. Monographs and Studies in Mathematics, 19. Pitman (Advanced Publishing Program), Boston, MA, 1983. xi+356 pp. ISBN: 0-273-08600-6

For Further Reading IV

- Strichartz, Robert, A guide to distribution theory and Fourier transforms. Reprint of the 1994 original [CRC, Boca Raton; MR1276724]. World Scientific Publishing Co., Inc., River Edge, NJ, 2003. x+226 pp. ISBN: 981-238-430-8
- Taylor, Michael E. Pseudodifferential operators. Princeton Mathematical Series, 34. Princeton University Press, Princeton, N.J., 1981. xi+452 pp. ISBN: 0-691-08282-0

References to the work in the talk:

- E.T. Quinto, Tomographic reconstructions from incomplete data-numerical inversion of the exterior Radon transform, Inverse Problems 4(1988), 867-876.
- E.T. Quinto, *SIAM J. Math. Anal.* **24**(1993), 1215-1225.

For Further Reading V

- Characterization and reduction of artifacts in limited angle tomography, joint with Jürgen Frikel, Inverse Problems, 29 (2013) 125007 (21 pages). See also http://iopscience.iop.org/0266-5611/labtalk-article/55769
- Artifacts in incomplete data tomography with applications to photoacoustic tomography and sonar, joint with Jürgen Frikel, SIAM J. Appl. Math., 75(2),(2015) 703-725. (23 pages) Preprint on arXiv: http://arxiv.org/abs/1407.3453.
- Limited data problems for the generalized Radon transform in $\mathbb{R}^n$, joint with Jürgen Frikel, SIAM J. Math. Anal., 48(4)(2016), 2301-2318, Preprint on arXiv: http://arxiv.org/abs/1510.07151.

For Further Reading VI

- Analyzing Reconstruction Artifacts from Arbitrary Incomplete X-ray CT Data, L. Borg, J. Frikel, J. Jørgensen, E.T. Quinto. SIAM J. Imaging Sci., 11(4), 2786-2814 Oct. 2018. (29 pages)
- Microlocal analysis of a Compton tomography problem, James Webber TQ, arXiv: https://arxiv.org/abs/1902.09623, submitted, 2019.