Time Reversal Methods for Thermoacoustic Tomography

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CONFERENCE ON MODERN CHALLENGES IN IMAGING
IN THE FOOTSTEPS OF ALLAN MACLEOD CORMACK
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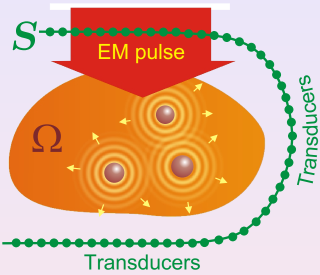
Outline

Thermoacoustic Tomography (TAT)



- Microwave ⇒ Thermoelastic Expansion ⇒ (Ultrasonic) Pressure Wave.
- Pressure measured on observation surface S in a time period $(0, T)$.
- One needs to find the initial pressure $f(x)$.

Mathematical Model

Wave propagation:

$$\begin{cases} u_{tt}(x,t) - c^2(x)\Delta u(x,t) = 0, \; x \in \mathbb{R}^n, \; t \geq 0, \\ u(x,0) = f(x), \; u_t(x,0) = 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Data: $g = u|_{S\times(0,T)}$.

Problem: Find f from g.

Techniques

1. Exact inversion formulas: Norton & Linzer ('80), Finch, Patch & Rakesh ('04), Xu & Wang ('05), Finch, Haltmeier & Rakesh ('07), Kunyansky ('07, '11,'15), N. ('09), Palamodov ('11 & 14), Natterer ('12), Salman ('12), Haltmeier ('11 & '13) . . .
2. Series solutions: Agranovsky & Kuchment ('07), Kunyansky ('07) . . .
3. Time-reversal: Finch, Patch & Rakesh ('04), Hristova, Kuchment & N. ('08), Hristova ('09), Stefanov & Uhlmann ('09), J. Qian, Stefanov, G. Uhlmann & H. Zhao ('11) . . .
4. Iterative methods: Huang et al. ('13), Belhachmi, Glatz, & Scherzer ('15), Arridge, et. al. (2016), Haltmeier & Nguyen ('18) . . .

Time Reversal: Main idea

Recall

$$\begin{cases} u_{tt}(x,t) - c^2(x)\,\Delta u(x,t) = 0, \ (x,t) \in \mathbb{R}^n \times (0,T), \\ u(x,0) = f(x), \ u_t(x,0) = 0, \quad x \in \mathbb{R}^n, \\ u(x,t) = g(x,t), \ (x,t) \in S \times (0,T). \end{cases}$$

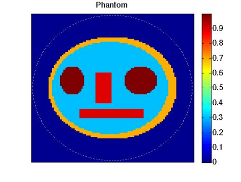
Assumption: S is a closed surface, Ω - domain bounded by S.
Crucial observation: The energy

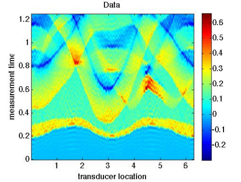
$$E_\Omega(u,t) := \int_\Omega c^{-2}|u_t|^2 + |\nabla u|^2 \, dx$$

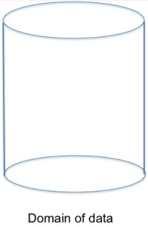
decays to zero as $t \to \infty$. Namely, $u \approx 0$ and $u_t \approx 0$ in Ω when t is big enough.

Simulated Data

Data example in 2D, closed observation surface:







Time Reversal: Simple/Effective Approach

Hristova, Kuchment, N. (IP '08) [1]: Time reversed problem

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, & x \in \Omega, \\ v(x,t) = \chi(t) g(x,t), & (x,t) \in S \times (0,T). \end{cases}$$

Then,

$$v(x,0) \approx f(x).$$

Here, $\chi \equiv 1$ for $t \in [0, T-\epsilon]$ and $\chi(T) = 0$.
Hristova (IP '09): rigorous proof for the approximation $v(.,0) \to f$ as $T \to \infty$.

[1] For constant speed Finch, Patch, & Rakesh ('04).

Finite/Short Time Approach

Stefanov& Uhlmann (IP '09), Qian, Stefanov, Uhlmann, Zhao (SIIMS '11)

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, \quad (x,t) \in \Omega \times (0,T), \\ v(x,T) = \phi(x), \quad v_t(x,T) = 0, \quad x \in \Omega, \\ v(x,t) = g(x,t), \quad (x,t) \in S \times (0,T), \end{cases}$$

where, ϕ is the harmonic extension of $g(\cdot, T)$ to Ω.
Define $A_T g = P_{H_0^1(\Omega)} v(\cdot, 0)$ - first order approximation of f.

Theorem

Under the visibility condition

$$\|f - A_T g\|_{H_0^1(\Omega)} \leq (1-\delta)\|f\|_{H_0^1(\Omega)},$$

for $0 < \delta < 1$.

Neumann Series Solution

- Since the error operator $\mathcal{E} : f \to f - A_T g$ is a contraction, one can repeat the time-reversal method to obtain the exact there construction. Formally

$$f = \sum_{i=0}^{\infty} \mathcal{E}^i A_T g.$$

  The Neumann series solution.
- Visibility condition: $T > \frac{1}{2} T(\Omega)$. That is, all *singularities* of f are observed on S.

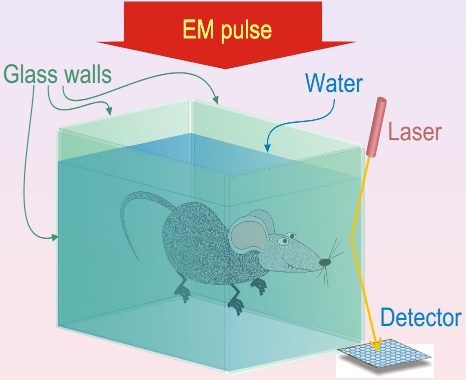
Outlook

The idea of time reversal and Neumann series solutions is efficient for TAT. We will see how it works in:

- TAT in a reverberant cavity
- TAT in elastic media

Experiment setup

UCL group - Ellwood, Zhang, Beard, Cox (JBO ’14):



Mathematical Model

Mathematical model:

$$\begin{cases} u_{tt}(x,t) - c^2(x)\Delta u(x,t) = 0, \ (x,t) \in \Omega \times (0,T), \\ u(x,0) = f(x), \ u_t(x,0) = 0, \quad x \in \Omega, \\ \partial_\nu u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

Data: $g = u|_{S \times (0,T)}$, where $S \subset \partial\Omega$.

Problem: Find f given g.

Difficulty: The energy $E_\Omega(u,t)$ is conserved for all time t.

Historical Remarks

- Kunyansky, Holman, & Cox (IP '13): series expansion (for constant speed and rectangular domain)
- Holman & Kunyansky (IP '15): gradual time reversal
- Stefanov & Yang (IP '15): averaged time reversal
- Acosta & Montalto (IP '15): conjugate gradient method
- Kunyansky & Nguyen (SIIMS '16): dissipative time-reversal

- Stefanov & Yang (IPI'17): Landweber's iteration
- Chervova & Oksanen(IP '16): dissipative time-reversal

Dissipative Time Reversal

Time reversed problem

$$\begin{cases} v_{tt}(x,t) - c^2(x)\,\Delta v(x,t) = 0, \; (x,t) \in \Omega \times (0,T), \\ v(x,T) = 0, \quad v_t(x,T) = 0, \quad x \in \Omega, \\ \partial_n v(x,t) - \lambda(x)\, v_t(x,t) = -\lambda(x)\, g_t(x,t), (x,t) \in \partial\Omega \times [0,T]. \end{cases}$$

Define:

$$A_T\, g = P_{H_0^1(\Omega)} v(.,0).$$

NOTE:

- $\lambda \equiv 1$ if $S = \partial\Omega$. Otherwise, a nonnegative smooth approximation of χ_S.
- u also the satisfies the same boundary condition

$$\partial_n u(x,t) - \lambda(x)\, u_t(x,t) = -\lambda(x)\, g_t(x,t), (x,t) \in \partial\Omega \times [0,T].$$

Observability Time

Observability time $T(S)$: LU of length of generalized geodesics rays until they hit S.

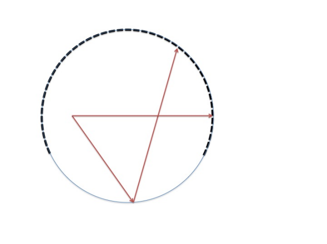


Figure: Generalized geodesics rays constant speed

Note: if $S = \partial\Omega$, $T(S) = T(\Omega)$.

Main Results

Assume Geometric Control Condition: $T(S) < T$.

Theorem (N.& Kunyansky (SIIMS '16))

1. *(Small time formula) We have*

$$\|f - A_T g\|_{H_0^1(\Omega)} \leq (1-\epsilon)\|f\|_{H_0^1(\Omega)}.$$

Consequentially, for $\mathcal{E} : f \to f - A_T g$

$$f = \sum_{m=0}^{\infty} \mathcal{E}^m A_T g.$$

2. *(Large time formula)* $\|f - A_T g\|_{H_0^1(\Omega)} \to 0$ *exponentially as* $T \to \infty$.

Proof

Error $e = u - v$,

$$\begin{cases} e_{tt}(x,t) - c^2(x)\,\Delta e(x,t) = 0, \ (x,t) \in \Omega \times (0,T), \\ e(x,T) = u(x,T), \quad e_t(x,T) = u_t(x,T), \quad x \in \Omega, \\ \partial_n e(x,t) - \lambda(x,t)\, e_t(x,t) = 0, (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

We have

$$E_\Omega(e,T) - E_\Omega(e,0) = \iint\limits_{\partial\Omega\times(0,T)} \lambda(x,t)\,|e_t(x,t)|^2\,dx\,dt.$$

Energy of e leaks out going backward in time.

Error Estimate

Bardos, Lebeau & Rauch (SICON '92) Under geometric control condition

$$\iint_{\partial\Omega\times(0,T)} \lambda(x,t)\,|e_t(x,t)|^2\,dx\,dt \geq \epsilon\, E_\Omega(e,T).$$

Then,

$$E_\Omega(e,0) \leq (1-\epsilon)E_\Omega(e,T) = (1-\epsilon)E_\Omega(u,T),$$

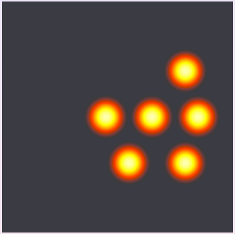
which implies

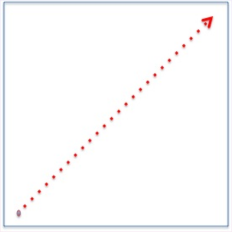
$$\|f - A_T g\|^2_{H_0^1(\Omega)} \leq (1-\epsilon)\|f\|^2_{H_0^1(\Omega)}.$$

This finishes the proof.

NOTE: the condition $T > T(S)$ does not seem to be optimal.

Numerical Setup: Full Data



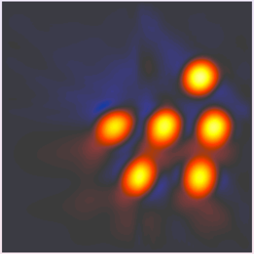


Numerical setup:

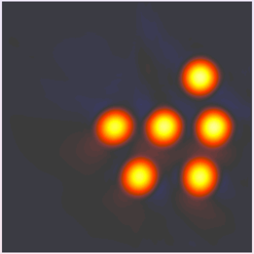
- Domain Ω is a square of size 2×2.
- Sound speed $c = 1$, $T(S) = T(\Omega) = 2\sqrt{2}$.

Iterative Reconstruction

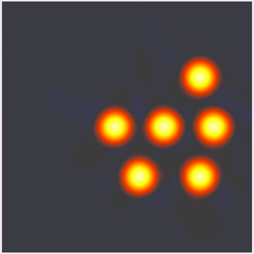
Neumann series solution for $T = 1.6 > \frac{1}{2}T(S)$.



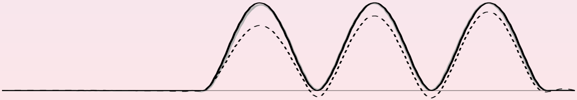
Iteration #1



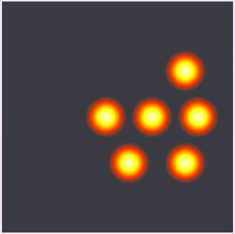
Iteration #2

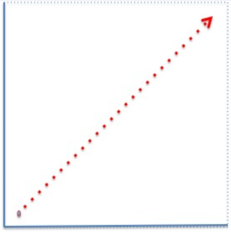


Iteration #5



Numerical Setup: Partial Data



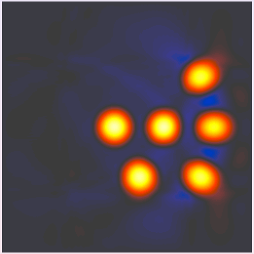


Numerical setup:

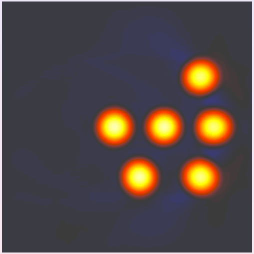
- Data collected left and lower edges
- $T(S) = 4\sqrt{2}$.

Iterative Reconstruction

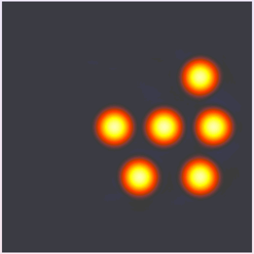
Reconstruction for partial data with short time $T = 3 > \frac{1}{2} T(S)$.



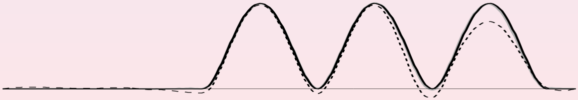
Iteration #1



Iteration #2

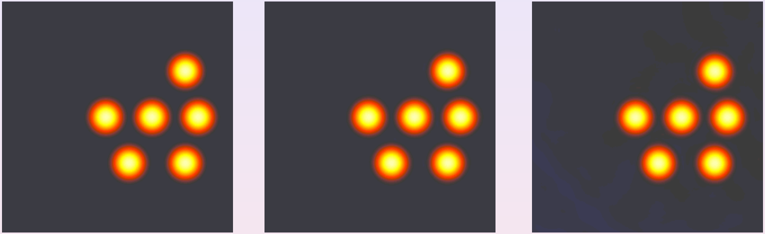


Iteration #5



Long-Time Reconstruction

Reconstruction with large time: $T = 5$.



Phantom | From full data | From partial data

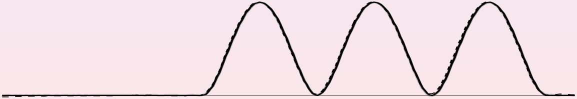
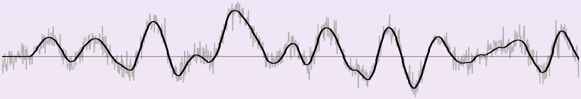


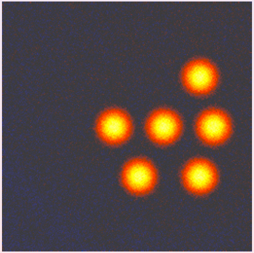
Figure: Gray: phantom, dashed: partial data, black: full data

Noisy data

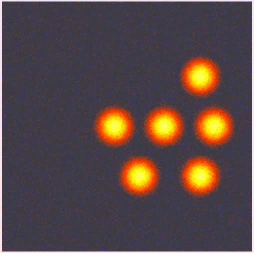
Reconstruction with large time: $T = 5$. Noisy data: 50% noise (in L^2 norm).



Phantom



From partial data



From full data

Central Line Profile

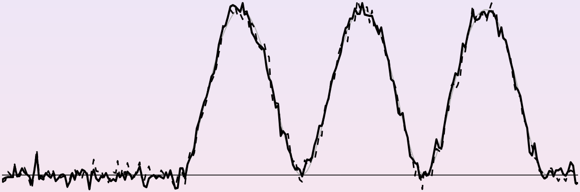


Figure: Central line profile

TAT in Elastic Media

Wave propagation:

$$\begin{cases} \mathbf{u}_{tt}(x,t) - \Delta^* \mathbf{u}(x,t) = 0, \; x \in \mathbb{R}^n, \; t \geq 0, \\ \mathbf{u}(x,0) = \mathbf{f}(x), \; \mathbf{u}_t(x,0) = 0, \quad x \in \mathbb{R}^n. \end{cases}$$

Here,

- $\Delta^* \mathbf{u} = \nabla \cdot [\mu(x)(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)] + \nabla(\lambda(x) \nabla \cdot \mathbf{u})$
- λ, μ Lamé parameters
- Jacobian

$$(\nabla \mathbf{u})_{i,j} = \frac{\partial \mathbf{u}_i}{\partial x_j}.$$

Data: $\mathbf{g} = \mathbf{u}|_{S \times (0,T)}$.

Problem: Find $\mathbf{f}$ from $\mathbf{g}$. That is, to invert $\Lambda : \mathbf{f} \to \mathbf{g}$.

A Bit About Elastic Waves

Two modes of propagation:

- P-wave (longitudinal/fast wave): propagate with speed $c_p(x) = \sqrt{\lambda(x) + 2\mu(x)}$
- S-wave (shear/slow wave): propagate with speed $c_s(x) = \sqrt{\mu(x)}$.

Time Reversal

Tittelfitz (IP '11)

$$\begin{cases} \mathbf{v}_{tt}(x,t) - \Delta^* \mathbf{v}(x,t) = 0, & (x,t) \in \Omega \times (0,T), \\ \mathbf{v}(x,T) = \phi(x), \quad \mathbf{v}_t(x,T) = 0, & x \in \Omega, \\ \mathbf{v}(x,t) = \mathbf{g}(x,t), & (x,t) \in S \times (0,T). \end{cases}$$

Denote

$$A_T \mathbf{g} = P_{H_0^1(\Omega)} \mathbf{v}(0)$$

time reversal reconstruction of f. Then,

$$\|\mathbf{f} - A_T \mathbf{g}\|_{H_0^1(\Omega)} \le (1-\delta) \|\mathbf{f}\|_{H_0^1(\Omega)}$$

if $c_p(x) < 3c_s(x)$ and T is big enough.

Our Result

Katsnelson & N. (AML '17)

Theorem

Under the visibility condition:

$$\|\mathbf{f} - A_T \mathbf{g}\|_{H_0^1(\Omega)} \leq (1-\delta)\|\mathbf{f}\|_{H_0^1(\Omega)}.$$

Visibility condition:

$$T > \frac{1}{2} \max\{T_s(\Omega), T_p(\Omega)\}.$$

That is, all the singularities are observed on S.

Proof

Need to prove escaped energy $E_{\Omega^c}(\mathbf{u}, T)$ is not negligible

$$\|\mathbf{f}\|^2_{H^1_0(\Omega)} \leq C E_{\Omega^c}(\mathbf{u}, T).$$

Indeed, write $E_{\Omega^c}(\mathbf{u}, T) := \|(\mathbf{u}(T)|_{\Omega^c}, \mathbf{u}_t(T)|_{\Omega^c})\|^2$, then

- Microlocal analysis estimate

$$\|\mathbf{f}\|^2_{H^1_0(\Omega)} \leq c(\|(\mathbf{u}(T)|_{\Omega^c}, \mathbf{u}_t(T)|_{\Omega^c})\|^2 + \|\mathbf{u}(T)\|^2_{L^2(\Omega)})$$

- Injectivity of the mapping $\mathbf{f} \to (\mathbf{u}(T)|_{\Omega^c}, \mathbf{u}_t(T)|_{\Omega^c})$

A theorem by Taylor finishes the proof.

Proof of Injectivity

Two ingredients - Tittelfitz(SIAP '15):

Theorem (Domain of Dependence)

Let $c_+ = \sup_x c_p(x)$. *Assume that* $\mathbf{u}(\cdot, t_0) = 0$ *for all* $x \in B_{c_+(t_1 - t_0)}(x_0)$ *then* $\mathbf{u} = 0$ *on*

$$\{(x,t) : t_0 \le t \le t_1, x \in B_{c_+(t_1-t)}(x_0).\}$$

Theorem (Unique Continuation)

Assume that $\mathbf{u} = 0$ *on a neighborhood of* x_0 *on* (t_1, t_2). *Then* $\mathbf{u}(x,t) = 0$ *for* (x,t) *in the double cone*

$$\left\{(x,t) : dist_s(x, x_0) + \left| t - \frac{t_1 + t_2}{2} \right| \le \frac{t_2 - t_1}{2} \right\}.$$

Domain of Depend. vs Unique Cont.

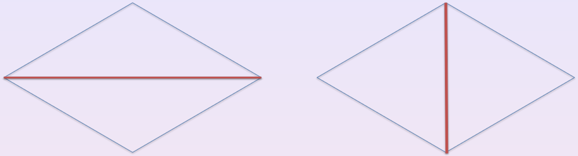


Figure: Acoustic Case: $u_{tt} - c^2 u_{xx} = 0$, same slope $\pm\frac{1}{c}$

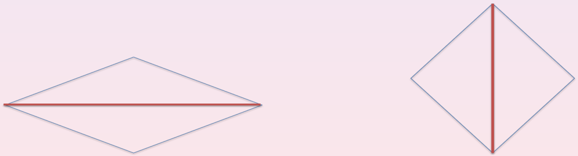
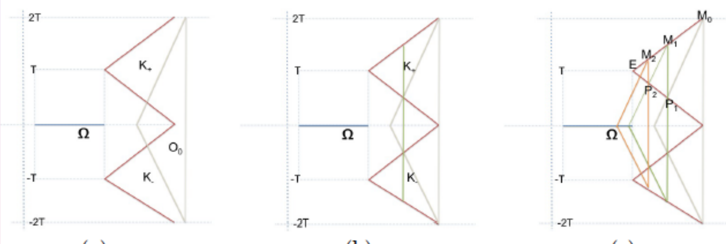


Figure: Elastic case: different slopes: $\pm\frac{1}{c_p}$ (left) and $\pm\frac{1}{c_s}$ (right).

Proof Done!

Assume that $\mathbf{u}(T)|_{\Omega^c} \equiv \mathbf{u}_t(T)|_{\Omega^c} \equiv 0$. Extend the solution $\mathbf{u}$ evenly. Then, domain peeling argument follows to move to $\partial\Omega$.



(a) (b) (c)

Unique continuation and visibility condition shows $\mathbf{f} \equiv 0$.

Questions

Thank you for your attention!