The X-ray transform on constant curvature disks

François Monard

Dept. of Mathematics, University of California Santa Cruz

August 6, 2019

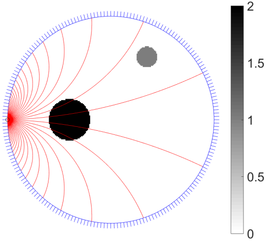
MS. 3: Gen. RT and Applications in Imaging

Cormack Conference, Tufts University

Toy Model: The X-ray transform on CCD's

Let M the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1,1)$ define the metric $g_\kappa(z) := (1+\kappa|z|^2)^{-2}|dz|^2$ on M, of constant curvature 4κ.

$\kappa = -0.8$

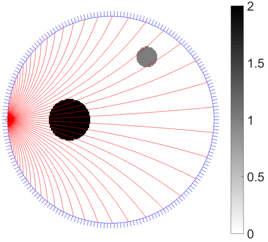


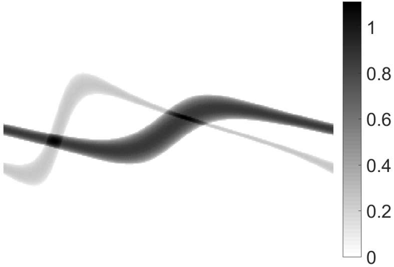
A family of **simple** metrics which degenerates at $\kappa \to \pm 1$.

Toy Model: The X-ray transform on CCD's

Let M the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1,1)$ define the metric $g_\kappa(z) := (1+\kappa|z|^2)^{-2}|dz|^2$ on M, of constant curvature 4κ.

$\kappa = -0.4$



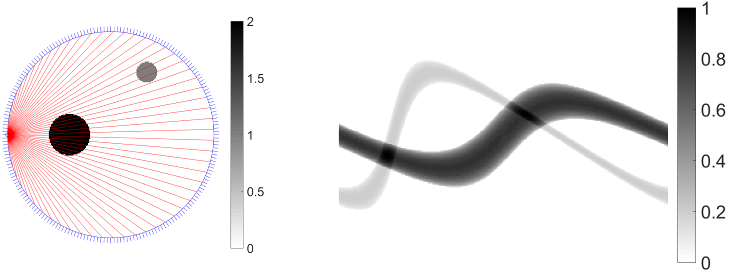


A family of **simple** metrics which degenerates at $\kappa \to \pm 1$.

Toy Model: The X-ray transform on CCD's

Let M the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1,1)$ define the metric $g_\kappa(z) := (1+\kappa|z|^2)^{-2}|dz|^2$ on M, of constant curvature 4κ.

$\kappa = 0$

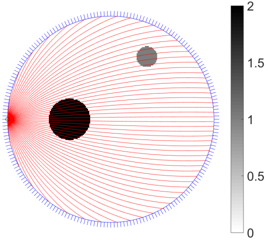


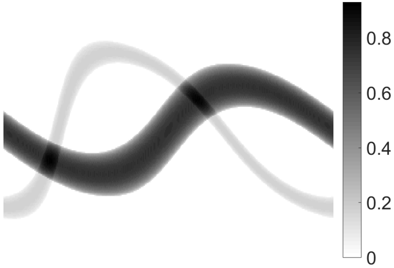
A family of **simple** metrics which degenerates at $\kappa \to \pm 1$.

Toy Model: The X-ray transform on CCD's

Let M the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1,1)$ define the metric $g_\kappa(z) := (1+\kappa|z|^2)^{-2}|dz|^2$ on M, of constant curvature 4κ.

$\kappa = 0.4$



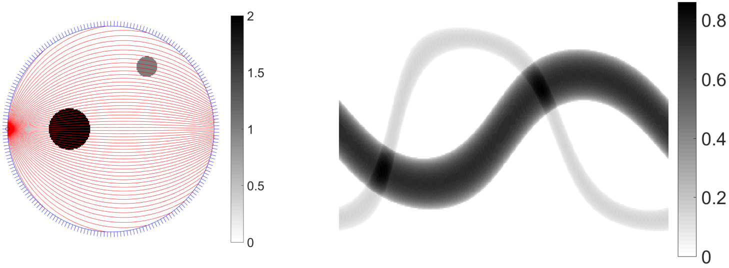


A family of **simple** metrics which degenerates at $\kappa \to \pm 1$.

Toy Model: The X-ray transform on CCD's

Let M the unit disk in $\mathbb{R}^2$. For $\kappa \in (-1,1)$ define the metric $g_\kappa(z) := (1+\kappa|z|^2)^{-2}|dz|^2$ on M, of constant curvature 4κ.

$\kappa = 0.8$



A family of **simple** metrics which degenerates at $\kappa \to \pm 1$.

The XRT on media with variable refractive index

The general project is to understand the XRT on manifolds.

Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc…

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible !)
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

The XRT on media with variable refractive index

The general project is to understand the XRT on manifolds. Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc...

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations. SVD (if possible !).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

The XRT on media with variable refractive index

The general project is to understand the XRT on simple manifolds. Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc...

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible !).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

The XRT on media with variable refractive index

The general project is to understand the XRT on simple surfaces.
Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc. . .

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible !).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

The XRT on media with variable refractive index

The general project is to understand the XRT on CCD's.
Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc...

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible !).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

The XRT on media with variable refractive index

The general project is to understand the XRT on CCD's.
Applications to

- X-ray CT in media with variable refractive index.
- Travel-time tomography/boundary rigidity, etc...

By 'understand' we mean:

- Injectivity. Stability estimates.
- Reconstruct various types of integrands (functions, vectors, tensor fields) explicitly and efficiently.
- Range characterizations, SVD (if possible !).
- Mitigate the trade-off between parallel and fan-beam geometries (starting with the Euclidean case).

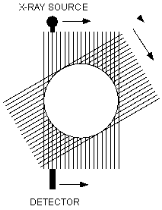
Parallel v/s fan-beam geometry

Parallel geometry: enjoys the Fourier Slice theorem, which allows for a rigorous, efficient regularization theory.

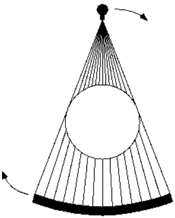
Fan-beam geometry:

- 'natural' acquisition geometry, then traditionally rebinned into parallel data before processed. [Natterer '01]
- no parallel geometry on non-homogeneous surfaces. Instead, PDE's on the unit phase space.

The **Euclidean disk** benefits from both viewpoints.



PARALLEL



FAN-BEAM

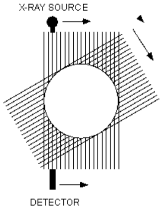
Parallel v/s fan-beam geometry

Parallel geometry: enjoys the Fourier Slice theorem, which allows for a rigorous, efficient regularization theory.

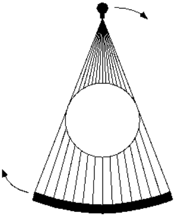
Fan-beam geometry:

- 'natural' acquisition geometry, then traditionally rebinned into parallel data before processed. [Natterer '01]
- no parallel geometry on non-homogeneous surfaces. Instead, PDE's on the unit phase space.

The **Euclidean disk** benefits from both viewpoints.



PARALLEL


FAN-BEAM

Literature, or rather, authors. . .

Radon, Funk, Helgason, Ludwig, Gel'fand, Graev, Quinto, Cormack, Natterer, Maass, Louis, Rigaud, Hahn, Kuchment, Agranovsky, Ambartsoumian, Krishnan, Abishek, Mishra

Herglotz, Wiechert, Zoeppritz, Anikonov, Romanov, Mukhometov,

Sharafutdinov, Pestov, Uhlmann, Vasy, Stefanov, Zhou, Assylbekov,

Paternain, Salo, Ilmavirta, Guillarmou, Guillemin, Railo,

Lehtonen, Cekić...

Literature, or rather, authors. . .

Radon, Funk, Helgason, Ludwig, Gel'fand, Graev, Quinto, Cormack, Natterer, Maass, Louis, Rigaud, Hahn, Kuchment, Agranovsky, Ambartsoumian, Krishnan, Abishek, Mishra

Herglotz, Wiechert, Zoeppritz, Anikonov, Romanov, Mukhometov,

Sharafutdinov, Pestov, Uhlmann, Vasy, Stefanov, Zhou, Assylbekov,

Paternain, Salo, Ilmavirta, Guillarmou, Guillemin, Railo,

Lehtonen, Cekić. . .

PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 83, Number 2, October 1981

THE RADON TRANSFORM ON A FAMILY OF CURVES IN THE PLANE[1]

A. M. CORMACK

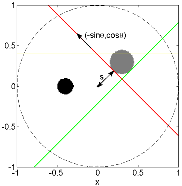
ABSTRACT. Inversion formulas are given for Radon's problem when the line integrals are evaluated along curves given, for a fixed (p, ϕ), by $r^{\alpha} \cos|\alpha(\theta - \phi)| = p^{\alpha}$, where α is real, $\alpha \neq 0$.

Outline

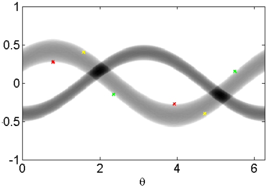
The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}\times\mathbb{S}^1)$

$$\mathcal{R}f(s,\theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^\perp + t\hat{\theta})\ dt, \quad (s,\theta)\in\mathbb{R}\times\mathbb{S}^1.$$



$\leftrightarrow$



Moment conditions: Gelfand, Graev, Helgason, Ludwig

$\mathcal{D}(s,\theta) = \mathcal{R}f(s,\theta)$ for some f iff

(i) $\mathcal{D}(s,\theta) = \mathcal{D}(-s,\theta+\pi)$ for all $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$.

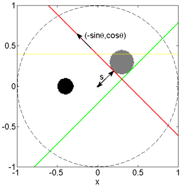
(ii) For $k\ge 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s,\theta)\ ds = \sum_{\ell=-k}^{k} a_{\ell,k} e^{ik\theta}$.

$\Leftrightarrow \int_{\mathbb{S}^1}\int_{\mathbb{R}} \mathcal{D}(s,\theta) s^k e^{ip\theta}\ ds\ d\theta = 0$, $|p|>k$, $p+k$ even

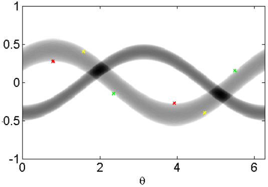
The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}\times\mathbb{S}^1)$

$$\mathcal{R}f(s,\theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^\perp + t\hat{\theta})\ dt, \quad (s,\theta)\in\mathbb{R}\times\mathbb{S}^1.$$



$\leftrightarrow$



Moment conditions: Gelfand, Graev, Helgason, Ludwig

$\mathcal{D}(s,\theta) = \mathcal{R}f(s,\theta)$ for some f iff

(i) $\mathcal{D}(s,\theta) = \mathcal{D}(-s,\theta+\pi)$ for all $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$.

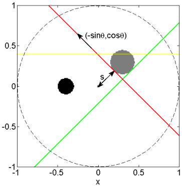
(ii) For $k\ge 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k\mathcal{D}(s,\theta)\ ds = \sum_{\ell=-k}^k a_{\ell,k}e^{ik\theta}$.

$\Leftrightarrow$ $\int_{\mathbb{S}^1}\int_{\mathbb{R}} \mathcal{D}(s,\theta)s^k e^{ip\theta}\ ds\ d\theta = 0, \ |p|>k, \ p-k$ even.

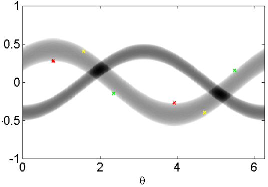
The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R}\times\mathbb{S}^1)$

$$\mathcal{R}f(s,\theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^\perp + t\hat{\theta})\ dt, \quad (s,\theta)\in\mathbb{R}\times\mathbb{S}^1.$$



$\leftrightarrow$



Moment conditions: Gelfand, Graev, Helgason, Ludwig

$$\mathcal{D}(s,\theta) = \mathcal{R}f(s,\theta) \text{ for some } f \text{ iff}$$

(i) $\mathcal{D}(s,\theta) = \mathcal{D}(-s,\theta+\pi)$ for all $(s,\theta)\in\mathbb{R}\times\mathbb{S}^1$.

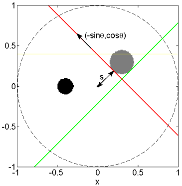
(ii) For $k\ge 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s,\theta)\ ds = \sum_{\ell=-k}^{k} a_{\ell,k} e^{ik\theta}$.

$\Leftrightarrow \int_{\mathbb{S}^1}\int_{\mathbb{R}} \mathcal{D}(s,\theta) s^k e^{ip\theta}\ ds\ d\theta = 0,\ |p|>k,\ p-k$ even.

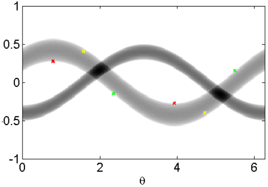
The classical moment conditions

Parallel geometry: $\mathcal{R} : \mathcal{S}(\mathbb{R}^2) \to \mathcal{S}(\mathbb{R} \times \mathbb{S}^1)$

$$\mathcal{R}f(s,\theta) = \int_{\mathbb{R}} f(-s\hat{\theta}^\perp + t\hat{\theta})\ dt, \quad (s,\theta) \in \mathbb{R} \times \mathbb{S}^1.$$



$\leftrightarrow$



Moment conditions: Gelfand, Graev, Helgason, Ludwig

$$\mathcal{D}(s,\theta) = \mathcal{R}f(s,\theta) \text{ for some } f \text{ iff}$$

(i) $\mathcal{D}(s,\theta) = \mathcal{D}(-s,\theta+\pi)$ for all $(s,\theta) \in \mathbb{R} \times \mathbb{S}^1$.

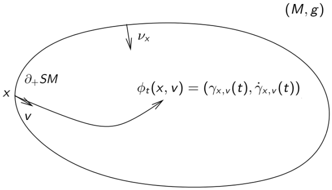
(ii) For $k \ge 0$, $p_k(\theta) := \int_{\mathbb{R}} s^k \mathcal{D}(s,\theta)\ ds = \sum_{\ell=-k}^{k} a_{\ell,k} e^{ik\theta}$.

$\Leftrightarrow$ $\int_{\mathbb{S}^1} \int_{\mathbb{R}} \mathcal{D}(s,\theta) s^k e^{ip\theta}\ ds\ d\theta = 0$, $|p| > k$, $p-k$ even.

The Pestov-Uhlmann range characterization

$I_0 : C^\infty(M) \to C^\infty_+(\partial_+ SM)$

$$I_0 f(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt.$$



$\mathcal{S}$: scattering relation

Range characterization of I_0:

$$\boxed{I_0(C^\infty(M)) = P_-(C^\infty_\alpha(\partial_+ SM)),}$$ [Pestov-Uhlmann '05]

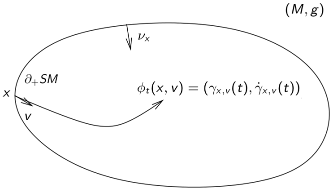
P_- takes the form $P_- := A_-^* H_- A_+$, where

- $A_+ : C^\infty(\partial_+ SM) \to C^\infty(\partial SM)$ symmetrization w.r.t. $\mathcal{S}$.
- H_-: odd Hilbert transform on the fibers of ∂SM.
- $A_-^* : C^\infty(\partial SM) \to C^\infty(\partial SM)$: $A_-^* f(x,v) = f(x,v) - f(\mathcal{S}(x,v))$.

The Pestov-Uhlmann range characterization

$I_0 : C^\infty(M) \to C_+^\infty(\partial_+ SM)$

$$I_0 f(x,v) = \int_0^{\tau(x,v)} f(\gamma_{x,v}(t))\, dt.$$



$\mathcal{S}$: scattering relation

Range characterization of I_0:

$$\boxed{I_0(C^\infty(M)) = P_-(C_\alpha^\infty(\partial_+ SM)),}$$

[Pestov-Uhlmann '05]

P_- takes the form $P_- := A_-^* H_- A_+$, where

- $A_+ : C^\infty(\partial_+ SM) \to C^\infty(\partial SM)$ symmetrization w.r.t. $\mathcal{S}$.
- H_-: odd Hilbert transform on the fibers of ∂SM.
- $A_-^* : C^\infty(\partial SM) \to C^\infty(\partial SM)$: $A_-^* f(x,v) = f(x,v) - f(\mathcal{S}(x,v))$.

Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".

We also understand how the cokernel can be realized by other types of integrands.

Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".

We also understand how the cokernel can be realized by other types of integrands.

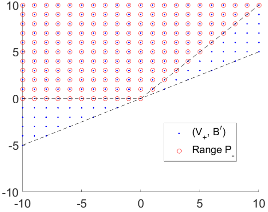
Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".



$$u'_{p,q} = e^{ip\beta}\left(e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha}\right)$$

We also understand how the cokernel can be realized by other types of integrands.

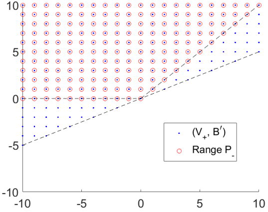
Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta,\alpha) = (\beta+\pi+2\alpha, \pi-\alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".



$$u'_{p,q} = e^{ip\beta}\left(e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha}\right)$$

We also understand how the cokernel can be realized by other types of integrands.

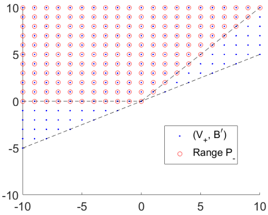
Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".



$$u'_{p,q} = e^{ip\beta}\left(e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha}\right)$$

We also understand how the cokernel can be realized by other types of integrands.

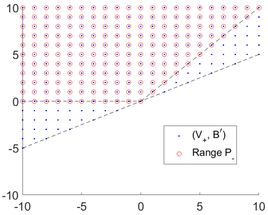
Equivalence of ranges characterizations

Theorem (M., IPI, '15)

Both range characterizations above are equivalent.

Sketch of proof: Understand the operator $P_- = A_-^* H_- A_+$.

- Euclidean scattering relation: $\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\alpha, \pi - \alpha)$.
- Explicit construction of the SVD of $P_- : L^2(\partial_+ SM) \to L^2(\partial_+ SM)$.
- Reparameterized moment conditions is equivalent to saying "$\mathcal{D} \perp$ Range P_-".

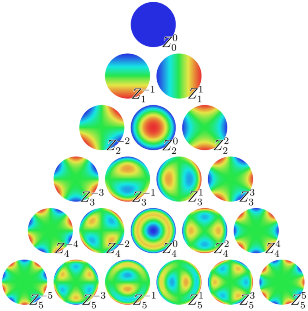


$$u'_{p,q} = e^{ip\beta}(e^{i(2q+1)\alpha} + (-1)^p e^{i(2(p-q)-1)\alpha})$$

We also understand how the cokernel can be realized by other types of integrands.

SVD of I_0

Zernike polynomials:
$Z^{n,k}$, $n \in \mathbb{N}_0$, $0 \le k \le n$.



Uniquely defined through the properties:

[Kazantzev-Bukhgeym '07]

- $Z_{n,0} = z^n$.
- $\partial_{\bar z} Z_{n,k} = -\partial_z Z_{n,k-1}$, $1 \le k \le n$.
- $Z_{n,k}|_{\partial M}(e^{i\beta}) = e^{i(n-2k)\beta}$.

In addition,

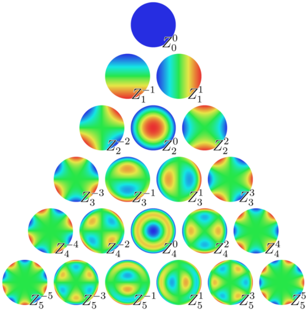
$$(Z_{n,k}, Z_{n',k'})_{L^2(M)} = \frac{\pi}{n+1}\, \delta_{n,n'}\, \delta_{k,k'}.$$

$$I_0[Z^{n,k}] = \frac{C}{n+1} e^{i(n-2k)(\beta+\alpha+\pi)} (e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}).$$

(in parallel coordinates, $\beta + \alpha + \pi = \theta$ and $\sin\alpha = s$)

SVD of I_0

Zernike polynomials:
$Z^{n,k}$, $n \in \mathbb{N}_0$, $0 \leq k \leq n$.



Uniquely defined through the properties:
[Kazantzev-Bukhgeym '07]

- $Z_{n,0} = z^n$.
- $\partial_{\bar z} Z_{n,k} = -\partial_z Z_{n,k-1}$, $1 \leq k \leq n$.
- $Z_{n,k}|_{\partial M}(e^{i\beta}) = e^{i(n-2k)\beta}$.

In addition,

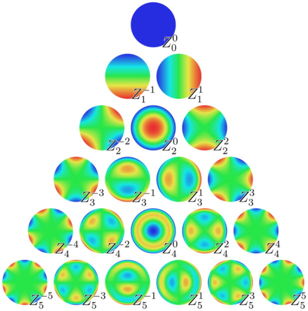
$$(Z_{n,k}, Z_{n',k'})_{L^2(M)} = \frac{\pi}{n+1}\, \delta_{n,n'}\, \delta_{k,k'}.$$

$$I_0[Z^{n,k}] = \frac{C}{n+1} e^{i(n-2k)(\beta+\alpha+\pi)}\left(e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}\right).$$

(in parallel coordinates, $\beta + \alpha + \pi = \theta$ and $\sin\alpha = s$)

SVD of I_0

Zernike polynomials: $Z^{n,k}$, $n \in \mathbb{N}_0$, $0 \le k \le n$.



Uniquely defined through the properties:

[Kazantzev-Bukhgeym '07]

- $Z_{n,0} = z^n$.
- $\partial_{\bar z} Z_{n,k} = -\partial_z Z_{n,k-1}$, $1 \le k \le n$.
- $Z_{n,k}|_{\partial M}(e^{i\beta}) = e^{i(n-2k)\beta}$.

In addition,

$$(Z_{n,k}, Z_{n',k'})_{L^2(M)} = \frac{\pi}{n+1}\,\delta_{n,n'}\,\delta_{k,k'}.$$

$$I_0[Z^{n,k}] = \frac{C}{n+1} e^{i(n-2k)(\beta+\alpha+\pi)}\left(e^{i(n+1)\alpha} + (-1)^n e^{-i(n+1)\alpha}\right).$$

(in parallel coordinates, $\beta + \alpha + \pi = \theta$ and $\sin\alpha = s$)

Outline

Range characterizations: statement

Theorem (Mishra-M., preprint '19)

Let M equipped with the metric g_κ for $\kappa \in (-1, 1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

Range characterizations: statement

Theorem (Mishra-M., preprint '19)

Let M equipped with the metric g_κ for $\kappa \in (-1,1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ *for some* $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. `[Pe-Uhl, '04]`
2. *u satisfies a complete set of othogonality/moment conditions:*

$$(u, \psi^\kappa_{n,k})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \qquad n \ge 0,\ k < 0,\ k > n,$$

where we have defined

$$\psi^\kappa_{n,k} := \frac{(-1)^n}{4\pi}\sqrt{\mathfrak{s}'_\kappa(\alpha)}\, e^{i(n-2k)(\beta+\mathfrak{s}_\kappa(\alpha))}\left(e^{i(n+1)\mathfrak{s}_\kappa(\alpha)} + (-1)^n e^{-i(n+1)\mathfrak{s}_\kappa(\alpha)}\right),$$

$$\mathfrak{s}_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa}\tan\alpha\right). \qquad (\mathfrak{s}_0(\alpha) = \alpha, \qquad \mathfrak{s}_\kappa \circ \mathfrak{s}_{-\kappa} = id)$$

3. $C_- u = 0$, *where* $C_- := \frac{1}{2} A_-^* H_- A_-$ $\qquad (id + C_-^2 = \mathbb{1}_{\mathrm{Ran}\, I_0})$

Range characterizations: statement

Theorem (Mishra-M., preprint '19)

Let M equipped with the metric g_κ for $\kappa \in (-1,1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ *for some* $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. [Pe-Uhl, '04]
2. u *satisfies a complete set of othogonality/moment conditions:*

$$(u, \psi^\kappa_{n,k})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \qquad n \ge 0,\ k < 0,\ k > n,$$

where we have defined

$$\psi^\kappa_{n,k} := \frac{(-1)^n}{4\pi} \sqrt{\mathfrak{s}'_\kappa(\alpha)}\, e^{i(n-2k)(\beta + \mathfrak{s}_\kappa(\alpha))} \left(e^{i(n+1)\mathfrak{s}_\kappa(\alpha)} + (-1)^n e^{-i(n+1)\mathfrak{s}_\kappa(\alpha)} \right),$$

$$\mathfrak{s}_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa} \tan\alpha \right). \qquad (\mathfrak{s}_0(\alpha) = \alpha, \qquad \mathfrak{s}_\kappa \circ \mathfrak{s}_{-\kappa} = id)$$

3. $C_- u = 0$, *where* $C_- := \frac{1}{2} A_-^* H_- A_-$

Range characterizations: statement

Theorem (Mishra-M., preprint ’19)

Let M equipped with the metric g_κ for $\kappa \in (-1,1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ *for some* $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. [Pe-Uhl, ’04]
2. u *satisfies a complete set of othogonality/moment conditions:*

$$(u, \psi^\kappa_{n,k})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \qquad n \ge 0,\ k < 0,\ k > n,$$

where we have defined

$$\psi^\kappa_{n,k} := \frac{(-1)^n}{4\pi}\sqrt{\mathfrak{s}'_\kappa(\alpha)}\, e^{i(n-2k)(\beta+\mathfrak{s}_\kappa(\alpha))}\left(e^{i(n+1)\mathfrak{s}_\kappa(\alpha)} + (-1)^n e^{-i(n+1)\mathfrak{s}_\kappa(\alpha)}\right),$$

$$\mathfrak{s}_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa}\tan\alpha\right). \qquad (\mathfrak{s}_0(\alpha) = \alpha, \qquad \mathfrak{s}_\kappa \circ \mathfrak{s}_{-\kappa} = id)$$

3. $C_- u = 0$, *where* $C_- := \frac{1}{2} A_-^* H_- A_-$ $\quad (id + C_-^2 = \Pi_{Ran\ I_0})$

Range characterizations: statement

Theorem (Mishra-M., preprint '19)

Let M equipped with the metric g_κ for $\kappa \in (-1,1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ *for some* $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. [Pe-Uhl, '04]
2. u *satisfies a complete set of othogonality/moment conditions:*

$$(u, \psi^\kappa_{n,k})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \qquad n \ge 0,\ k < 0,\ k > n,$$

where we have defined

$$\psi^\kappa_{n,k} := \frac{(-1)^n}{4\pi}\sqrt{\mathfrak{s}_\kappa'(\alpha)}\, e^{i(n-2k)(\beta + \mathfrak{s}_\kappa(\alpha))}\left(e^{i(n+1)\mathfrak{s}_\kappa(\alpha)} + (-1)^n e^{-i(n+1)\mathfrak{s}_\kappa(\alpha)}\right),$$

$$\mathfrak{s}_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa}\tan\alpha\right). \qquad (\mathfrak{s}_0(\alpha) = \alpha, \qquad \mathfrak{s}_\kappa \circ \mathfrak{s}_{-\kappa} = id)$$

3. $C_- u = 0$, *where* $C_- := \frac{1}{2} A_-^* H_- A_-$ $\quad (id + C_-^2 = \Pi_{Ran\ I_0})$

Range characterizations: statement

Theorem (Mishra-M., preprint '19)

Let M equipped with the metric g_κ for $\kappa \in (-1,1)$. Suppose $u \in C^\infty(\partial_+ SM)$ such that $S_A^* u = u$. Then $u \in I_0(C^\infty(M))$ iff either of the following is satisfied

1. $u = P_- w$ for some $w \in C^\infty_{\alpha,+,-}(\partial_+ SM)$. [Pe-Uhl, '04]
2. u satisfies a complete set of othogonality/moment conditions:

$$(u, \psi^\kappa_{n,k})_{L^2(\partial_+ SM, d\Sigma^2)} = 0, \qquad n \ge 0,\ k < 0,\ k > n,$$

where we have defined

$$\psi^\kappa_{n,k} := \frac{(-1)^n}{4\pi} \sqrt{\mathfrak{s}'_\kappa(\alpha)} e^{i(n-2k)(\beta + s_\kappa(\alpha))} \left(e^{i(n+1)\mathfrak{s}_\kappa(\alpha)} + (-1)^n e^{-i(n+1)\mathfrak{s}_\kappa(\alpha)} \right),$$

$$\mathfrak{s}_\kappa(\alpha) := \tan^{-1}\left(\frac{1-\kappa}{1+\kappa} \tan\alpha \right). \qquad (\mathfrak{s}_0(\alpha) = \alpha, \qquad \mathfrak{s}_\kappa \circ \mathfrak{s}_{-\kappa} = id)$$

3. $C_- u = 0$, where $C_- := \frac{1}{2} A_-^* H_- A_-$ $\qquad (id + C_-^2 = \Pi_{Ran\ I_0})$

Proof

In light of the first item, understand the action of $P_- = A_-^* H_- A_+$, e.g. find its SVD. Construct functions that

- extend smoothly under $A_\pm$
- transform well under **fiberwise Hilbert transform** and **scattering relation**

$$\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\mathfrak{s}_\kappa(\alpha), \pi - \alpha).$$

- are even or odd w.r.t. $\mathcal{S}_A := \mathcal{S} \circ (\alpha \mapsto \alpha + \pi)$

This produces four families of functions, some giving the $L^2 - L^2$ SVD of P_- and the eigendecomposition of C_-.
In particular, Ran $P_- = \ker C_-$.
The SVD picture is identical to the Euclidean one !

Proof

In light of the first item, understand the action of $P_- = A_-^* H_- A_+$, e.g. find its SVD. Construct functions that

- extend smoothly under $A_\pm$
- transform well under **fiberwise Hilbert transform** and **scattering relation**

$$\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\mathfrak{s}_\kappa(\alpha), \pi - \alpha).$$

- are even or odd w.r.t. $\mathcal{S}_A := \mathcal{S} \circ (\alpha \mapsto \alpha + \pi)$

This produces four families of functions, some giving the $L^2 - L^2$ SVD of P_- and the eigendecomposition of C_-.
In particular, Ran $P_- = \ker C_-$.
The SVD picture is identical to the Euclidean one !

Proof

In light of the first item, understand the action of $P_- = A_-^* H_- A_+$, e.g. find its SVD. Construct functions that

- extend smoothly under $A_\pm$
- transform well under **fiberwise Hilbert transform** and **scattering relation**

$$\mathcal{S}(\beta, \alpha) = (\beta + \pi + 2\mathfrak{s}_\kappa(\alpha), \pi - \alpha).$$

- are even or odd w.r.t. $\mathcal{S}_A := \mathcal{S} \circ (\alpha \mapsto \alpha + \pi)$

This produces four families of functions, some giving the $L^2 - L^2$ SVD of P_- and the eigendecomposition of C_-.
In particular, Ran $P_- =$ ker C_-.
The SVD picture is identical to the Euclidean one !

Outline

SVD: Statement

Theorem (Mishra-M., preprint '19)

Let M be the unit disk equipped with the metric g_κ for $\kappa \in (-1,1)$. Define $\mathfrak{s}_\kappa(\alpha)$ and $\{\psi^\kappa_{n,k}\}_{n\ge 0, k\in\mathbb{Z}}$ as above, as well as

$$\widehat{Z^\kappa_{n,k}}(z) := \sqrt{\frac{n+1}{\pi}}(1-\kappa)\frac{1+\kappa|z|^2}{1-\kappa|z|^2}\underbrace{Z_{n,k}}_{Zernike}\left(\frac{1-\kappa}{1-\kappa|z|^2}z\right),$$

$$\widehat{\psi^\kappa_{n,k}} := 2\sqrt{1+\kappa}\ \psi^\kappa_{n,k}, \qquad \sigma^\kappa_{n,k} := \frac{1}{\sqrt{1-\kappa}}\frac{2\sqrt{\pi}}{\sqrt{n+1}}.$$

- $\{\widehat{Z^\kappa_{n,k}}\}_{n\ge 0,\ 0\le k\le n}$ *ONB of* $L^2(M, w_\kappa)$ *where* $w_\kappa(z) := \frac{1+\kappa|z|^2}{1-\kappa|z|^2}$.
- $\{\widehat{\psi^\kappa_{n,k}}\}_{n\ge 0,\ 0\le k\le n}$ *ON in* $L^2(\partial_+ SM, d\Sigma^2) \cap \ker(id - \mathcal{S}_A^*)$.

For any $f \in w_\kappa L^2(M, w_\kappa)$ *expanding as*

$$f = w_\kappa \sum_{n\ge 0}\sum_{k=0}^{n} f_{n,k}\widehat{Z^\kappa_{n,k}}, \qquad \text{we have} \qquad I_0 f = \sum_{n\ge 0}\sum_{k=0}^{n}\sigma^\kappa_{n,k}\ f_{n,k}\ \widehat{\psi^\kappa_{n,k}}.$$

Proof (sketch)

- Take the functions in the range of I_0, namely,

$$\psi^{\kappa}_{n,k}, \qquad n \geq 0, \qquad 0 \leq k \leq n,$$

and prove that $Z^{\kappa}_{n,k} := I_0^* \psi^{\kappa}_{n,k}$ is orthogonal on M for some weight [Maass, Louis].

- Also show that $I_0^* \psi^{\kappa}_{n,k} = 0$ for $k \notin 0 \ldots n$.

Note: I_0^* depends on the weight in data space. Since $\psi^{\kappa}_{n,k}$ is orthogonal in $L^2(\partial_+ SM)$, it is natural to define I_0^* w.r.t. this topology.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)}\, d\theta$$

$$\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\, \mathfrak{s}'(\alpha_-)\, d\theta \qquad (U_n : \text{Cheb 2})$$

Note the following relation:

$$\beta_-(\rho,\theta) + \mathfrak{s}(\alpha_-(\rho,\theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1+\kappa\rho^2\cos(2\theta)}\right) = \theta'(\rho,\theta)$$

Now change variable $\theta \to \theta'$ with $\frac{\partial\theta'}{\partial\theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\, \mathfrak{s}'(\alpha_-(\rho,\theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1+\kappa\rho^2}{1-\kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2}\rho\sin\theta'\right)\, d\theta'$$

Now use Euclidean knowledge.

At least 3 miracles along the way.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)}\, d\theta$$

$$\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\, \mathfrak{s}'(\alpha_-)\, d\theta \qquad (U_n : \text{Cheb 2})$$

Note the following relation:

$$\beta_-(\rho,\theta) + \mathfrak{s}(\alpha_-(\rho,\theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1 + \kappa\rho^2\cos(2\theta)} \right) = \theta'(\rho,\theta)$$

Now change variable $\theta \to \theta'$ with $\frac{\partial\theta'}{\partial\theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\, \mathfrak{s}'(\alpha_-(\rho,\theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1+\kappa\rho^2}{1-\kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2} \rho \sin\theta' \right)\, d\theta'$$

Now use Euclidean knowledge.

At least 3 miracles along the way.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)}\ d\theta$$

$$\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\ \mathfrak{s}'(\alpha_-)\ d\theta \qquad (U_n : \text{Cheb } 2)$$

Note the following relation:

$$\beta_-(\rho,\theta) + \mathfrak{s}(\alpha_-(\rho,\theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1+\kappa\rho^2\cos(2\theta)}\right) = \theta'(\rho,\theta)$$

Now change variable $\theta \to \theta'$ with $\frac{\partial\theta'}{\partial\theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\ \mathfrak{s}'(\alpha_-(\rho,\theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1+\kappa\rho^2}{1-\kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2}\rho\sin\theta'\right)\ d\theta'$$

Now use Euclidean knowledge.

At least 3 miracles along the way.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$\begin{aligned} I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) &\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)}\ d\theta \\ &\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\ \mathfrak{s}'(\alpha_-)\ d\theta \qquad (U_n : \text{Cheb } 2) \end{aligned}$$

Note the following relation:

$$\beta_-(\rho, \theta) + \mathfrak{s}(\alpha_-(\rho, \theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1 + \kappa\rho^2 \cos(2\theta)}\right) = \theta'(\rho, \theta)$$

Now change variable $\theta \to \theta'$ with $\frac{\partial\theta'}{\partial\theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\ \mathfrak{s}'(\alpha_-(\rho, \theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1+\kappa\rho^2}{1-\kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2}\rho \sin\theta'\right)\ d\theta'$$

Now use Euclidean knowledge.

At least 3 miracles along the way.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)}\, d\theta$$

$$\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\, \mathfrak{s}'(\alpha_-)\, d\theta \qquad (U_n : \text{Cheb 2})$$

Note the following relation:

$$\beta_-(\rho,\theta) + \mathfrak{s}(\alpha_-(\rho,\theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1 + \kappa\rho^2\cos(2\theta)}\right) = \theta'(\rho,\theta)$$

Now change variable $\theta \to \theta'$ with $\frac{\partial \theta'}{\partial \theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\ \mathfrak{s}'(\alpha_-(\rho,\theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1+\kappa\rho^2}{1-\kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2}\rho\sin\theta'\right)\ d\theta'$$

Now use Euclidean knowledge.

At least 3 miracles along the way.

Proof (ugly)

$(\beta_-, \alpha_-)(\rho, \theta)$: coordinates of the unique curve through $(\rho e^{i0}, \theta)$.

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} \sqrt{\mathfrak{s}'(\alpha_-)} \frac{e^{i(n+1)\mathfrak{s}(\alpha_-)} + (-1)^n e^{-i(n+1)\mathfrak{s}(\alpha_-)}}{2\cos(\alpha_-)} \, d\theta$$

$$\propto e^{i(n-2k)\omega} \int_{\mathbb{S}^1} e^{i(n-2k)(\beta_- + \mathfrak{s}(\alpha_-))} U_n(\sin(\mathfrak{s}(\alpha_-)))\ \mathfrak{s}'(\alpha_-) \, d\theta \qquad (U_n : \text{Cheb } 2)$$

Note the following relation:

$$\beta_-(\rho, \theta) + \mathfrak{s}(\alpha_-(\rho, \theta)) + \pi = \theta - \tan^{-1}\left(\frac{\kappa\rho^2 \sin(2\theta)}{1 + \kappa\rho^2 \cos(2\theta)} \right) = \theta'(\rho, \theta)$$

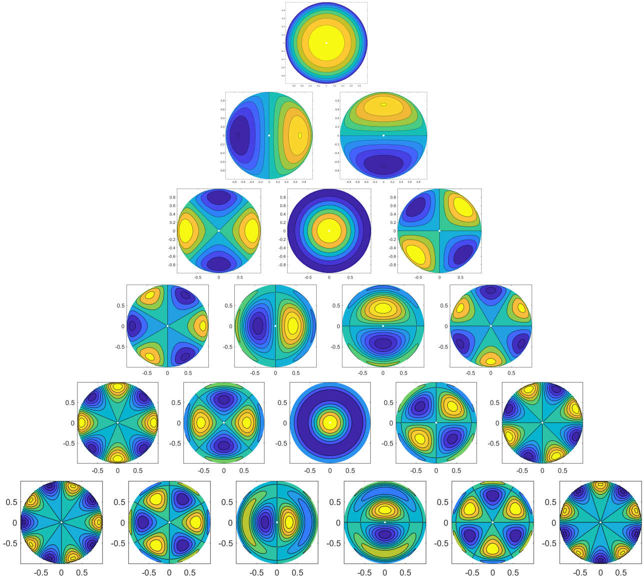
Now change variable $\theta \to \theta'$ with $\frac{\partial \theta'}{\partial \theta} \propto \frac{1-\kappa\rho^2}{1+\kappa\rho^2}\ \mathfrak{s}'(\alpha_-(\rho, \theta))$:

$$I_0^* \psi_{n,k}^\kappa(\rho e^{i\omega}) \propto e^{i(n-2k)\omega} \frac{1 + \kappa\rho^2}{1 - \kappa\rho^2} \int_{\mathbb{S}^1} e^{i(n-2k)\theta'} U_n\left(-\frac{1-\kappa}{1-\kappa\rho^2} \rho \sin\theta' \right) \, d\theta'$$

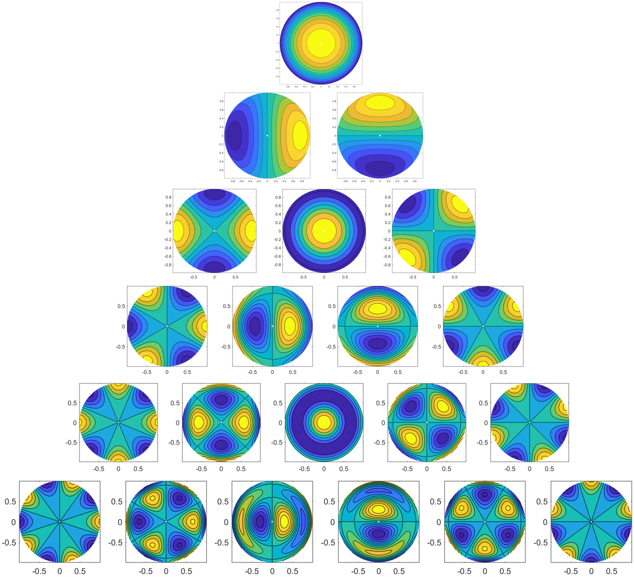
Now use Euclidean knowledge.

At least 3 miracles along the way.

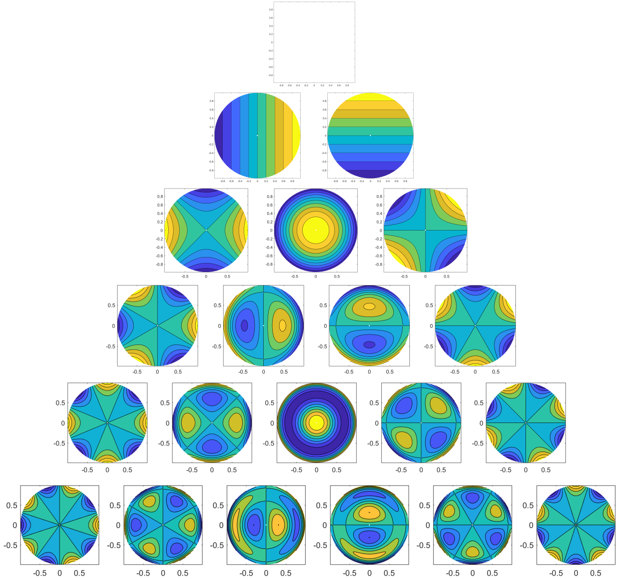
Visualization: $\{Z_{n,k}^{\kappa}\}_{0\le n\le 5, 0\le k\le n}$, $\kappa = -0.8$



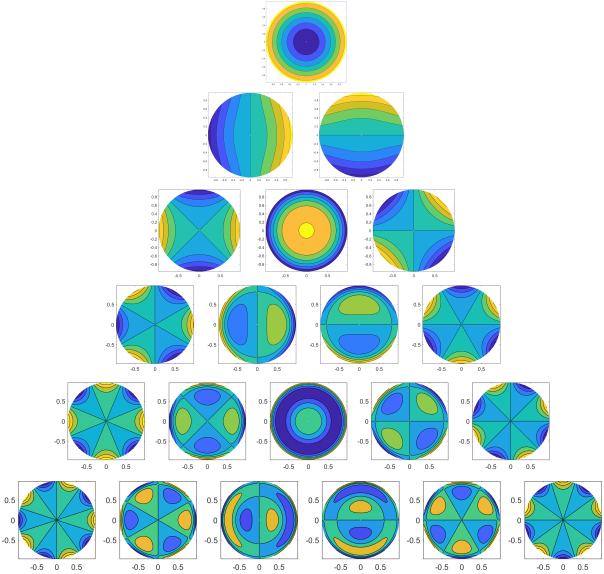
Visualization: $\{Z^{\kappa}_{n,k}\}_{0\le n\le 5, 0\le k\le n}$, $\kappa = -0.4$



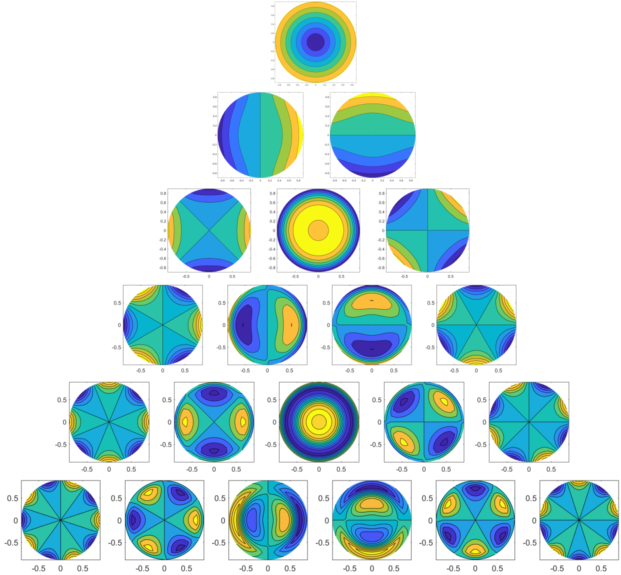
Visualization: $\{Z^{\kappa}_{n,k}\}_{0\le n\le 5, 0\le k\le n}$, $\kappa = 0$



Visualization: $\{Z_{n,k}^{\kappa}\}_{0\le n\le 5, 0\le k\le n}$, $\kappa = 0.4$



Visualization: $\{Z_{n,k}^{\kappa}\}_{0\le n\le 5, 0\le k\le n}$, $\kappa = 0.8$



Conclusions

On the geodesic X-ray transform on constant curvature disks

- Range characterizations via either projection operators or moment conditions.
- SVD of I_0 for a special choice of weights on M and $\partial_+ SM$.

Perspectives:

- tensor tomography, regularity of special invariant distributions,
- sharp Sobolev mapping properties for I_0.
- generalize to other (non-CC, non-symmetric) geometries.

Thank you

Reference:

- R. K. Mishra, F. M., *Range characterizations and Singular Value Decomposition of the geodesic X-ray transform on disks of constant curvature*, preprint (2019) - arXiv:1906.09389

Conclusions

On the geodesic X-ray transform on constant curvature disks

- Range characterizations via either projection operators or moment conditions.
- SVD of I_0 for a special choice of weights on M and $\partial_+ SM$.

Perspectives:

- tensor tomography, regularity of special invariant distributions,
- sharp Sobolev mapping properties for I_0.
- generalize to other (non-CC, non-symmetric) geometries.

Thank you

Reference:

- R. K. Mishra, F. M., *Range characterizations and Singular Value Decomposition of the geodesic X-ray transform on disks of constant curvature*, preprint (2019) - arXiv:1906.09389

Conclusions

On the geodesic X-ray transform on constant curvature disks

- Range characterizations via either projection operators or moment conditions.
- SVD of I_0 for a special choice of weights on M and $\partial_+ SM$.

Perspectives:

- tensor tomography, regularity of special invariant distributions,
- sharp Sobolev mapping properties for I_0.
- generalize to other (non-CC, non-symmetric) geometries.

Thank you

Reference:

- R. K. Mishra, F. M., *Range characterizations and Singular Value Decomposition of the geodesic X-ray transform on disks of constant curvature*, preprint (2019) - arXiv:1906.09389