# Histogram Tomography 

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Aside: In my home town of Whaley Bridge, Derbyshire over this weekend


## Bragg edge spectra



Each crystolagraphic plane produces one "edge". When the crystal is strained in the direction of the neutron beam the edge will move proportionately. The average shift of the mid point of the derivative of the spectrum gives the longitudinal ray transform of the strain. See talks by Uhlamnn, Kishnan, Monard, Abhishek, Vashisth in this meeting for more tensor tomography!

## Beyond integrals

In conventional tomography we consider integrals along ray in direction $\xi$

$$
R(x, \xi)=\int_{-\infty}^{\infty} f(x+t \xi) d t
$$

What if instead of the integral we knew the distribution or for discrete measurement to histogram along rays.
Put simply for each $x, \xi$, and each $y$ in the range of $f$ the measure of the set of $t$ values for which $f(x+t \xi)<y$ is the cumulative distribution $\Phi_{f, x, \xi}(y)$, and the distribution is $\phi_{f, x, \xi}(y)=\Phi_{f, x, \xi}^{\prime}(y)$.

## Illustration of distribution and cumulative distribution





We define the scalar histotomography transform

$$
H f(x, \xi)=\phi_{f, x, \xi}
$$

## Moments

For a function of one variable $f$

$$
\bar{f}=\int_{\mathbb{R}} y \phi_{f}(y) \mathrm{d} y=\int_{\Omega} f(x) \mathrm{d} x
$$

is the integral (rather than the mean in probability distributions).
We also define the $k$-th moment for $k \in \mathbb{N}$ as

$$
m_{k} f=\int_{\mathbb{R}} y^{k} \phi_{f}(y) \mathrm{d} y=\int_{\Omega} f(x)^{k} \mathrm{~d} x
$$

While the above is perhaps familiar for non-negative functions [2, Cor. A.1.] proves it for functions that can take negative values.

## Moment transform

We recover the Radon transform from the first moment of the Histotomography transform

$$
R f(x, \xi)=\overline{H f(x, \xi, \cdot)}
$$

We also have higher moments

$$
m_{k} H f(x, \xi, \cdot)=R\left(f^{k}\right)(x, \xi)
$$

For the scalar case each of the moments produce no more data than the Radon transform and for a non-negative function exactly the same data!
A non-negative bounded function is determined completely by its moments [1] so the data are identical.
It is interesting to note however that while fitting a function $f$ to its histotomography data Hf is a non-linear problem, each of the problems is linear

## Level set reconstruction

Consider the sub level set (ie inside a contour)

$$
\operatorname{Sf}(y):=\left\{x \in \mathbb{R}^{n} \mid f(x) \leq y\right\}
$$

Let $\chi_{S f(y)}(x)$ be the function that is 1 for $x \in S f(y)$ and 0 elsewhere. Then

$$
R \chi_{S f(y)}(x, \xi)=\Phi_{f, x, \xi}(y)
$$

so the cumulative distribution gives the Radon transform of the sub level set (or the histogram bin gives a 'fat' contour)

## Why?

Example: In infra-red chemical species tomography (CST) suppose there is a species that absorbs strongly at one wavelength, and suppose that wavelength depends monotonically on frequency. The absorption spectrum then gives a histogram of the temperature along the line.
In CST it is widely thought that because they measure a function rather than a scalar along each line they should be able to do tomography with fewer projections.
What we know is that for a full set of projections we have the same data several times (each moment), and that with each bin of the histogram we essentially get a contour.

## Tensor ray transforms

Vector field $v$, second rank symmetric tensor field $f$ define the longitudinal ray transform (LRT) as

$$
\begin{aligned}
& \operatorname{Iv}(x, \xi)=\int_{-\infty}^{\infty} \xi \cdot v(x+t \xi) \mathrm{d} t \\
& I f(x, \xi)=\int_{-\infty}^{\infty} \xi \cdot f(x+t \xi) \cdot \xi \mathrm{d} t
\end{aligned}
$$

for $x, \xi \in \mathbb{R}^{n}, \xi \neq 0$ where $\cdot$ denotes contraction.
Similarly rank for $k$.

The LRT has a null space consisting of potential tensor fields. In the case of vector fields this is just the usual definition, $f=\nabla u$ for a scalar $u$.
Potential rank-2 tensor fields are those that can be expressed as $f=\left(\nabla u+\nabla u^{T}\right) / 2$ for some vector field $u$. In general, following [12], we define the operator $d$ from rank- $k$ to rank $k+1$ formed by differentiation and symmetrization.
For $n \geq 2$ there is an explicit reconstruction for $f$ from If of filtered back projection type, modulo this null space.

Let $P_{\xi}$ be the projection of a symmetric second rank tensor field on to the plane perpendicular to $\xi$, then the transverse ray transform is defined as

$$
J f(x, \xi)=\int_{-\infty}^{\infty} P_{\xi} f(x+t \xi) \mathrm{d} t .
$$

We consider the important case of dimension $n=3$. For a direction $\eta \in \mathbb{R}^{3}$,

$$
\eta \cdot J f(x, \xi) \cdot \eta=\int_{-\infty}^{\infty} \eta \cdot f(x+t \xi) \cdot \eta \mathrm{d} t
$$

so in any plane normal to $\eta$ this is simply the Radon transform of the component $\eta \cdot f \cdot \eta$. This means there is a simple reconstruction for six suitably chosen[9] directions $\eta$.
Both these problems have histotomography version, in which data is the distribution of $\xi \cdot f(x+t \xi) \cdot \xi$ or $P_{\xi} f(x+t \xi)$ respectively, along the ray $x+t \xi$.

## A special case

In the case of the TRT with histogram data, one special case is that we have the distribution of $\eta \cdot f \cdot \eta$ along lines on a plane normal to $\eta$. As this is a Radon transform we have reduced to the scalar histotomography problem for this component on this plane. This means we can use any of the limited data methods we have for the scalar histotomography problem

## Doppler velocimetry

As Schuster [11] explains Doppler velocity tomography data is already understood as the distribution of velocity components along a line, in the direction of a line.
What we call the HLRT of the velocity field. The first moment is typically used and of course this gives only the solenoidal part of the velocity leaving the potential part to be determined by other means.

## Potential part from the moment data

However consider the second moment which is the integral of $(v . \xi)^{2}$ along the line. Suppose the solenoidal part of $v$ has already been recovered from the first moment and subtracted from the data, so without loss of generality $v=d u$ for a scalar $u$. We notice the second moment is nothing but the LRT of the rank-2 tensor $d u \odot d u$, from [12] we know we can recover the Saint-Venant tensor, or equivalently the Kröner tensor [6]

$$
K_{m n}=\epsilon_{m i k} \epsilon_{n j \ell}\left(u_{, i} u_{, j}\right)_{k \ell}=\epsilon_{m i k} \epsilon_{n j \ell} u_{, i k} u_{, j \ell}
$$

where indices after commas denote differentiation

## In 2D - a plane at a time

Consider now typical elements

$$
K_{11}=u_{, 23}^{2}-u_{, 22} u_{, 33}
$$

while

$$
K_{12}=2\left(u_{, 12} u_{, 33}-u_{, 13} u_{, 23}\right)
$$

and the tensor $K$ determines all the minors of the second derivative matrix $\left(d^{2} u\right)_{i j}=u_{, i j}$. Hence the adjugate matrix $\operatorname{Adj} d^{2} u$, and hence $d^{2} u$ up to sign if det non zero. In particular trace $d^{2} u=\nabla^{2} u$ is known and with suitable Dirichlet boundary data for $u$, we know $u$.

## If only we to do that for rank 2 !

The LRT is interesting as occurs in Bragg edge neutron strain tomography. Linear strain is a symmetric derivative $\epsilon=d u$ where $u$ is the displacement vector field.
This is in the null space of the $L R T$. In practice this means the LRT data can only measure the change in the shape of the exterior of an object.
Of course one can use the finite element method to find $u$ if the elastic modulus is known.
But can more data be extracted from neutron spectra?

## Bragg edge spectra



Each crystolagraphic plane produces one "edge". When the crystal is strained in the direction of the neutron beam the edge will move proportionately. The LRT comes from the average shift of the mid point of the derivative of the spectrum.

## A new insight

Zoom in on one Bragg edge (needs sufficient resolution of neutron wavelengths). The derivative of the spectrum gives the histogram LRT data up to a constant.


## Not so easy

Again from the histogram data we can deduce moments, and the $k$-th moment is the LRT of the symmetric powers $d u \odot \cdots \odot d u=$ $d u^{\odot k}$. These are not potential but they have a potential part. So we can recover the Saint Venant tensor of these.
Unfortunately each is a non-linear partial differential equation for
$u$. Solving for $u$ is more work than using the FEM method mentioned above.
However if the FEM method us used with assumed elastic moduli it is a way to check the solution is consistent with the full spectral data for each edge.

## Strain from diffraction pattern?

- Polycrystalline in a monochromatic x-rays beam give Bragg-Scherer rings [9]. Deformed to concentric ellipses by strain. Diffraction pattern in 2D but ellipses given by three parameters so not exactly histo-TRT.

- Single crystal diffraction (x-ray, electron, neutron?) gives diffraction spots blurred by strain, giving a distribution of displacement vectors. Hence histo-TRT. See eg [7]

More on histogram tomography in my preprint[8].

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