Mathematics arising from some recent imaging challenges

Peter Kuchment, Texas A & M University
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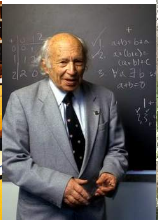
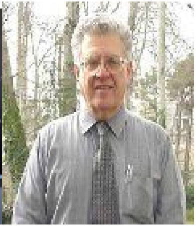
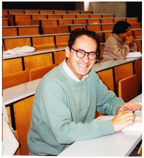
Modern Challenges in Imaging. In the Footsteps of Allan Cormack, on the 40th anniversary of his Nobel Prize
Tufts University, August 5 – 9, 2019.


GIANTS







Works mentioned in the lecture by ...

M. Agranovsky, G. Alberti, M. Allmaras, G. Amabartsoumian, H. Ammari, M. Anastasio, C. Appledorn, S. Arrige, G. Bal, P. Beard, E. Bonnetier, P. Burgholzer, Y. Capdeboscq, B. Cox, N. T. Do, R. Ehman, D. Finch, S. Gindikin, G. Gullberg, M. Haltmeier, K. Hickmann, Y. Hristova, A. Katsevich, P. Kuchment, L. Kunyansky, R. Kruger, V. Lin, A. Manduca, V. Markel, V. Maxim, J. McLaughlin, F. Monard, S. Moon, A. Nachman, L. Nguyen, M. Nguyen, A. Oraevsky, V. Palamodov, S. Patch, G. Patlauf, A. Pinkus, J. Qian, T. Quinto, Rakesh, K. Ren, O. Scherzer, J. Schotland, J. Seo, F. Steinhauer, P. Stefanov, A. Tamasan, F. Terzioglu, F. Triki, T. Truong, G. Uhlmann, L. Wang, H.-K. Zhao, T. Zhou ...

Apologies to the researchers, whose names I have unintentionally missed.

Surveys

Detailed references can be found in books and surveys, such as:

• G. Alberti and Y. Capdeboscq, *Lectures on elliptic methods for hybrid inverse problems*, Cours Spécialisés, v. 25, SMF, 2018.

• G. Bal, Hybrid inverse problems and internal functionals, in G. Uhlmann (Ed.), *Inside out*, V. 2, Cambridge Univ. Press 2013, pp. 325–368.

• P. Kuchment, Mathematics of Hybrid Imaging. A Brief Review, in *The Mathematical Legacy of Leon Ehrenpreis*, Springer, 2012, pp. 183 – 208.

• P. Kuchment, *The Radon Transform and Medical Imaging*, SIAM 2014

• F. Terzioglu, P. Kuchment, L. Kunyansky, Compton camera imaging and the cone transform: a brief overview, Inverse Problems, **34** (2018), 054002

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Resolution vs contrast controversy

There seem to be no cheap, safe, high contrast, and high resolution methods!

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An example: TAT/PAT - thermo/photo-acoustic tomography

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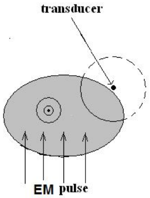

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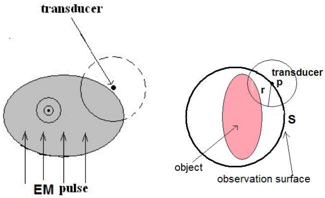

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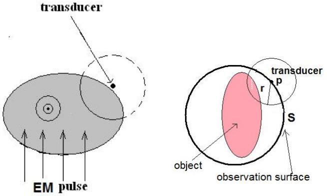

Recovering $f(x)$ from its restricted spherical means $M_S f(p, r)$.

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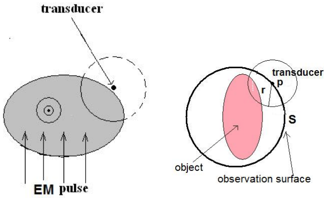

Recovering $f(x)$ from its restricted spherical means $M_S f(p, r)$.
Standard issues: uniqueness, inversion, stability, range, incomplete data effects. Mostly resolved, but **some gaps remain**.

The truth about TAT

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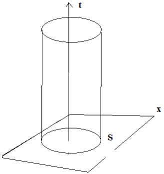
If $c(x)$ - sound speed, the model is:

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2(x)\Delta u, & \text{in } \mathbb{R}^3 \times \mathbb{R}_+ \\ u(0,x) = f(x), \frac{\partial u}{\partial t}(0,x) = 0 \end{cases}$$

Inversion of the observation operator



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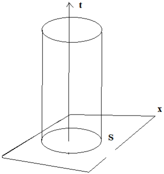

$f \mapsto g := u\,|_{S\times\mathbb{R}_+}$

Reduces to spherical means for constant sound speed only.
Related to spectral theory, transmission eigenvalues, spectral geometry, algebraic geometry, number theory.

Huygens' principle, energy decay, spectral theory relations

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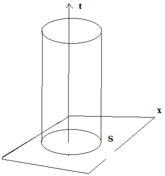

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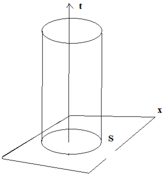

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- Huygens' principle and spectral theory. After time Fourier transform,

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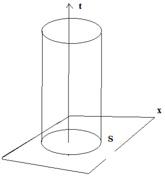

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Uniqueness sets? - largely open

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- **Time reversal** - all dimensions, stable, easy to implement, variable speed, any S, any location of f.

A TAT reconstruction

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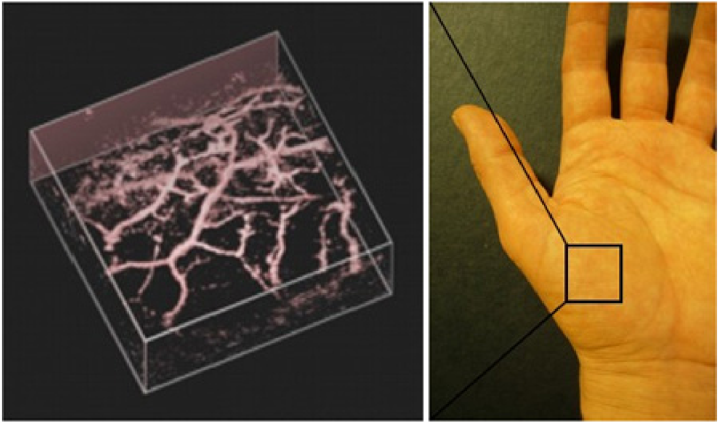

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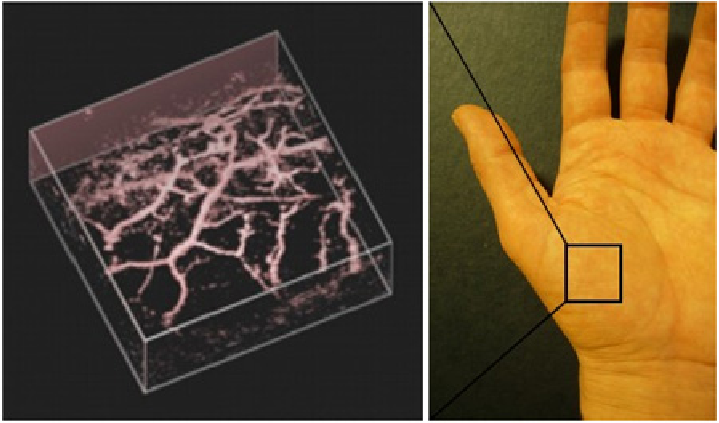

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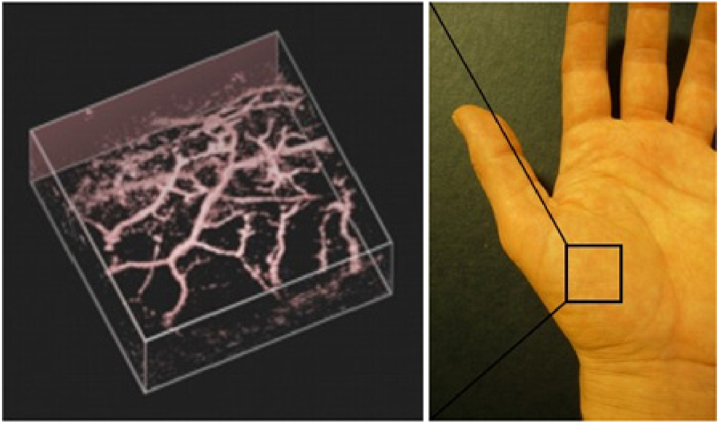

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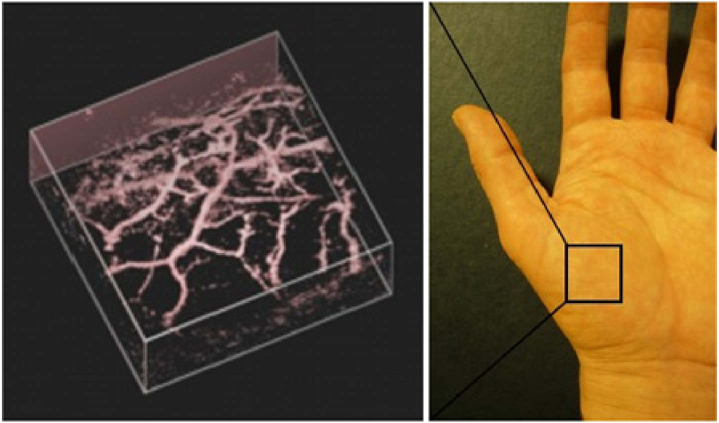

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Relations to the transmission eigenvalue problem

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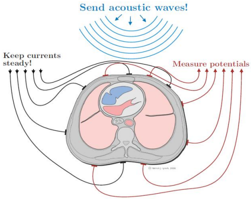
$$-\nabla \cdot D(x) \nabla I + a(x) I = 0$$

QPAT: An inverse problem with Internal data!

Other hybrid modalities that produce internal data. AET

Acousto Electric Tomography (AET)



Focusing the ultrasound.

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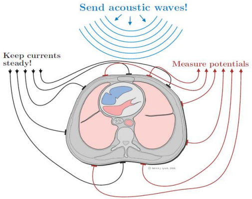

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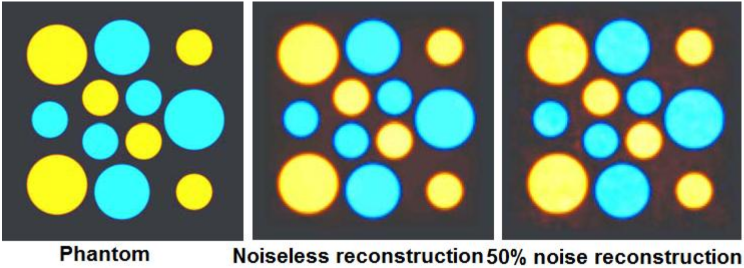
We measure - $\sigma(x)\nabla u_1(x) \cdot \nabla u_2(x)$. Internal information again!

AET - reconstructions



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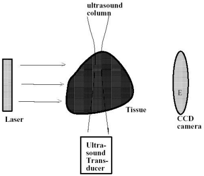

Diffusion/Absorption coefficients reconstruction.

Again not what we want: $G(x,d)A^2(x)I(x)$, A - ultrasound power, I light intensity, G - Green's function, d - detector position.

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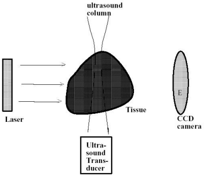

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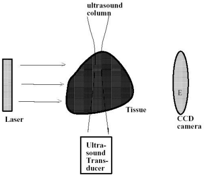

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And more: **MREIT, CDI, CDII, MRE, UE, ...**

Why does interior information stabilize?

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A: linearization and microlocal analysis of the resulting operator. A quick and reliable test of what is expected, but the proofs and the algorithms to implement the expectations are very involved.

Some “noisy” problems

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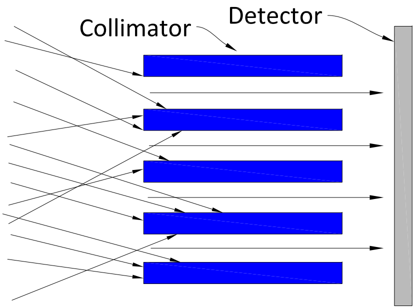
Issue: Low, up to extremely low (1%, .1%, or even less) SNR.

How is radiation detected?

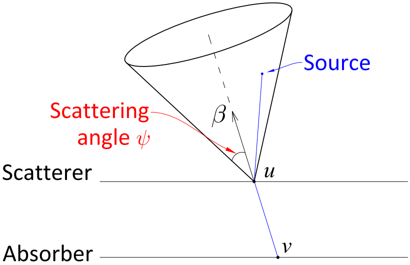
Standard collimated (Anger) γ-camera



Kills the signal when SNR is low!

Compton cameras & analogues

Compton γ-camera



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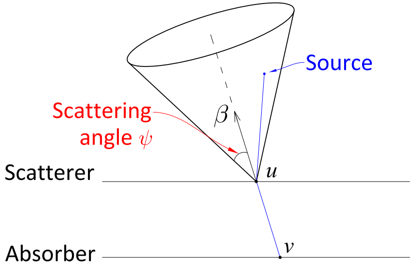

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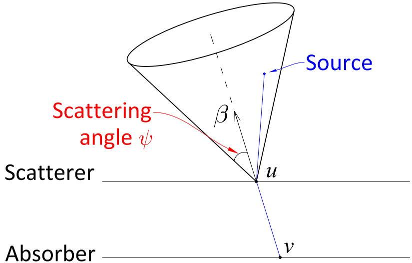

“Measures integrals over cones” (like in the Radon transform models, a lie when the counts are low).

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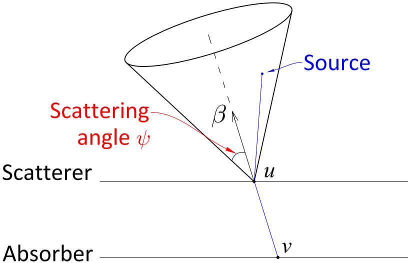

"Measures integrals over cones" (like in the Radon transform models, a lie when the counts are low). Analogs for neutron detectors.

Cone transforms, their properties and inversions

Overdetermined problem

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Overdetermined problem - GREAT!
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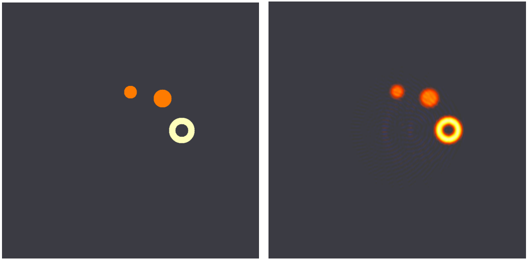
One has and must use ALL overdetermined data.A SPECT Compton data reconstruction (a 2D section of 3D reconstruction):



Beautiful (not fully completed) analysis: uniqueness, inversion, stability, microlocal, ...

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Also locates the source.

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See my mini-symposium talk for a little bit more and 2019 BIRS video for even more details.

The end


STOP
WRONG WAY


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TASHAKKUR ATU
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GRAZIE
MEHRBANI
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