# Theoretically exact solution of the inverse source problem for the wave equation with spatially and temporally reduced data. 

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## TAT/PAT set up

Thermo-acoustic tomography


PAT: uses laser beams (TAT uses radio frequency pulses).

Inverse source problem of TAT/PAT:
Find initial pressure from the measured pressure

## TAT/PAT mathematical model

Acoustic pressure $p(t, x)$ satisfies the wave equation Assume $c(x) \equiv 1$, no reflection, absorption, dispersion.

$$
\left\{\begin{array}{l}
p_{t t}=\Delta p, \quad x \in \mathbb{R}^{n} \\
p_{t}(0, x)=0, \quad p(0, x)=f(x)
\end{array}\right.
$$



Measurements $g(t, y) \equiv p(t, y)$ done on $S$.
TAT/PAT inverse source problem reconstructs $f(x)$ from $g(t, y)$.

## Solving the wave equation

Acoustic pressure $p(t, x)$ satisfies the wave equation

$$
\left\{\begin{array}{l}
p_{t t}=\Delta p, \quad x \in \mathbb{R}^{n} \\
p_{t}(0, x)=0, \quad p(0, x)=f(x)
\end{array}\right.
$$

Solution

$$
\begin{aligned}
p(t, y) & \equiv \frac{\partial}{\partial t} \int_{\Omega^{-}} f(x) \Phi_{n}(t, x-y) d x, \text { where } \\
\Phi_{2}(t, x) & =\frac{H(t-|x|)}{2 \pi \sqrt{t^{2}-|x|^{2}}}, \Phi_{3}(t, x)=\frac{\delta(t-|x|)}{4 \pi|x|}
\end{aligned}
$$

are Green functions of the free-space wave equation.
In particular, $g(t, y) \equiv p(t, y)$.

## Known inversion formulas for various surfaces $S$

$S$ is a plane: multiple works
"Universal formula" in 3D: a sphere, a plane, a cylinder (Xu \&
Wang)
Spheres (Finch et al; Kunyansky; Nguyen)
Ellipsoids and paraboloids (Natterer; Haltmeier; Palamodov;Salman)
Limiting cases of ellipsoids and paraboloids (Haltmeier \& Pereverzyev Jr.)
More complicated curves and surfaces (Palamodov)
Triangles, squares, cubes, and some tetrahedra (Kunyansky)
Corner-like domains in 3D, a segment of Coxeter cross in 2D
(Kunyansky)
Less explicit: series techniques (Kunyansky; Haltmeier et al)

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Less explicit: series techniques (Kunyansky; Haltmeier et al) All of these works either requires closed or unbounded $S$.

## Motivation

## Why reduced data?

(1) Limited data in space: practical reasons (bounded observation surface, detectors not surrounded from all sides)
(2) Truncated data in time: increase accuracy (reflection, scattering)

Simply applying known formulas on truncated data? Artifacts!
Approximate and iterative techniques exist.

## Our goal

To obtain explicit, theoretically exact inversion formulas that use temporally truncated data measured on open surface $S$.

## Our tools and approach

Radon projection of $f(x)$

$$
\mathcal{R} f(\omega, \tau)=\int_{\omega \cdot x=\tau} f(x) d x
$$


$\mathcal{R} f(\omega, \tau)=\mathcal{R} f(-\omega,-\tau)$

## Our tools and approach

Radon projection of $f(x)$

$$
\mathcal{R} f(\omega, \tau)=\int_{\omega \cdot x=\tau} f(x) d x
$$


$\mathcal{R} f(\omega, \tau)=\mathcal{R} f(-\omega,-\tau)$
A filtered backprojection inversion formula

$$
f(x)=\frac{1}{4 \pi} \mathcal{R}^{*} \mathcal{H} \frac{\partial}{\partial t} \mathcal{R} f
$$

where

$$
\begin{aligned}
& \left(\mathcal{R}^{*} h\right)(x) \equiv \int_{\mathbb{S}^{1}} h(x \cdot \omega, \omega) d \omega \text { - backprojection } \\
& (\mathcal{H} u)(p) \equiv \text { p.v. } \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(s)}{p-s} d s, \mathcal{H} \frac{\partial}{\partial t} \text { - filtration. }
\end{aligned}
$$

## Our tools and approach

Instead of $f(x)$ we reconstruct its Radon transform $\mathcal{R} f$

$$
\mathcal{R} f(\omega, \tau)=\int_{\omega \cdot x=\tau} f(x) d x=\int_{\Omega^{-}} f(x) \delta(-\omega \cdot x+\tau) d x
$$

We want to represent the plane wave $\delta$ as a retarded single layer potential.

$$
\delta(-\omega \cdot x+\tau)=\int_{T_{0}(\omega)}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-t, x-y) \varphi_{\omega}(t, y) d y d t
$$

using scattering theory!

## Layer potentials and scattering theory

Consider an incoming wave $u^{i n c}=u^{-}$. There will be a unique density $\varphi(t, y)$ defined on $\mathbb{R} \times \Gamma$ such that

$$
u^{ \pm}(\tau, x)=\int_{T_{0}(\omega)}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-t, x-y) \varphi(t, y) d y d t, x \in \Omega^{ \pm}
$$


where $u^{ \pm}$solves the wave equation and $u^{+}$solves the soft scattering problem, i.e. satisfies the jump conditions on 「

$$
\begin{aligned}
u^{+}(t, y) & =u^{-}(t, y) \\
\frac{\partial u^{-}(t, y)}{\partial n}-\frac{\partial u^{+}(t, y)}{\partial n} & =\varphi(t, y)
\end{aligned}
$$

## Scattering problem for plane wave

Now for $u^{\text {inc }}=\delta(\tau-\omega \cdot x)$ one obtains

$$
\delta(\tau-\omega \cdot x)=\int_{T_{0}(\omega)}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-t, x-y) \varphi_{\omega}(t, y) d y d t
$$

Plane wave $\delta$ is causal: $u^{\text {inc }}(\tau, x)=0$ if $\tau<\omega \cdot x$.
Due to the finite speed of propagation, both $u^{-}=u^{i n c}$, and $u^{+}=u^{\text {scat }}$ are 0 in front of the line $\omega \cdot x=t$.

Therefore, $\varphi_{\omega}(t, y)$ is also causal.

The sparse support of $\varphi_{\omega}(t, y)$ is crucial.


## Back to the inverse source problem

Measurements $g(t, y) \equiv p(t, y)$ on $S$ are given by:

$$
g(t, y)=\frac{\partial}{\partial t} G(t, y), G(t, y) \equiv \int_{\Omega^{-}} f(x) \Phi_{n}(t, x-y) d x, y \in S \subset \Gamma
$$

We want to recover the Radon projections of $f(x)$ defined as

$$
\mathcal{R} f(\omega, \tau) \equiv \int_{\Omega^{-}} f(x) \delta(\tau-\omega \cdot x) d x
$$

$$
\begin{aligned}
& \mathcal{R} f(\omega, \tau)=\int_{\Omega^{-}} f(x) \delta(-\omega \cdot x+\tau) d x \\
& =\int_{\Omega^{-}} f(x)\left[\int_{T_{0}}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-s, x-y) \varphi_{\omega}(s, y) d y d s\right] d x \\
& =\int_{0}^{\tau-T_{0}} \int_{\Gamma}\left[\int_{\Omega^{-}} f(x) \Phi_{n}(t, x-y) d x\right] \varphi_{\omega}(\tau-t, y) d y d t \\
& =\int_{0}^{\tau-T_{0}} \int_{\Gamma} G(t, y) \varphi_{\omega}(\tau-t, y) d y d t
\end{aligned}
$$

Do this for all $\omega \in \mathbb{S}^{n-1}, \tau \in \mathcal{T}(\omega)$, obtain $\mathcal{R} f(\omega, \tau)$.

## Spatially limited data

$$
\begin{aligned}
& \mathcal{R} f(\omega, \tau)=\int_{\Omega^{-}} f(x) \delta(-\omega \cdot x+\tau) d x \\
& =\int_{\Omega^{-}} f(x)\left[\int_{T_{0}}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-s, x-y) \varphi_{\omega}(s, y) d y d s\right] d x
\end{aligned}
$$



$$
\operatorname{supp}\left(\operatorname{Er}(\delta(\tau, \omega))=\bigcup_{y \in \Gamma \backslash S} \mathrm{~B}(y, \tau-\omega \cdot y)\right.
$$

## Inversion formula

Depending on $\omega$, there is interval of values $\tau$, for which equation

$$
\mathcal{R} f(\omega, \tau)=\int_{0}^{\tau-T_{0}} \int_{S} G(t, y) \varphi_{\omega}(\tau-t, y) d y d t
$$

is exact.
Theorem: For the following truncated acquisition geometry

we can reconstruct all Radon projections explicitly and exactly using temporally truncated data measured on an open surface $S$.

## Circular and spherical acquisition surfaces

Note that

$$
e^{i \rho \omega \cdot x}=\int_{\mathbb{R}} \delta(t-\omega \cdot x) e^{i \rho t} d t
$$

We represent $e^{i \rho \omega \cdot x}$ by a time harmonic single layer potential

$$
e^{i \rho \omega \cdot x}=\int_{\mathbb{S}^{n-1}} \widehat{\varphi_{\omega}}(\rho, \hat{y}) \hat{\Phi}_{n}(\rho, x-\hat{y}) d \hat{y}
$$

Take inverse Fourier transform, one obtain

$$
\delta(t-\omega \cdot x)=\int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}} \varphi_{\omega}(t, \hat{y}) \Phi_{n}(\tau-t, x-\hat{y}) d t d \hat{y}
$$

The densities are defined through their Fourier transforms.

## Circular geometry in 2D

Polar coord: $y=(R, \psi)$, and $\omega=(\cos \varpi, \sin \varpi)$
Define $\varphi_{\omega}$ through its Fourier transform $\hat{\varphi}_{\omega}$ :

$$
\hat{\varphi}_{\omega}(\rho, \hat{y}(\psi))= \begin{cases}\sum_{k=-\infty}^{\infty} \frac{2 i^{|k|} e^{i k(m-\psi)}}{\pi i H_{|k|}^{(1)}(\rho)}, & \rho \geq 0 \\ \frac{\hat{\varphi}_{\omega}(-\rho, \hat{y})}{}, & \rho<0\end{cases}
$$

## Truncated circular geometry

Consider the following truncated circular acquisition geometry:


Theorem: For the truncated circular geometry, formula

$$
\mathcal{R} f(\omega, \tau)=\int_{0}^{\tau-T_{0}} \int_{\mathbb{S}^{1}} \widetilde{G(t, y)} \varphi_{\omega}(\tau-t, y) d y d t
$$

holds for all $\omega \neq(0,-1)$ and $\tau$ lying within the intervals

$$
\tau \in \begin{cases}\left(-1,-\cos \left(\frac{\pi}{4}-\nu\right)+\sin \frac{\pi}{4}\right), & \nu \in\left(0, \frac{\pi}{2}\right], \\ \left(-1,-\cos \left(\frac{\pi}{4}+\nu\right)-\sin \frac{\pi}{4}\right), & \nu \in\left[\frac{\pi}{2}, \pi\right] .\end{cases}
$$

All Radon projections can be reconstructed exactly and explicitly from data measured on the open $S$ acquiring in a reduced temporal range of $[0,2-1 / \sqrt{2}] \approx[0,1.3]$ instead of the standard $[0,2]$.

## Simulation, circular geometry, 2D

Our phantom is a collection of slightly smoothed characteristic functions of circles.
$S$ is the acquisition surface.


Solve wave equation find $g(t, \omega(\theta+\pi))$


## Reconstruction results, truncated circular geometry



Exact $\operatorname{Rf}(\tau, \omega(\theta))$


Error w/o redundancy
Number of "detectors" = 512, reconstruction time $=0.4 \mathrm{sec}$.,


Reconstruction w/o redundancy


Final error
number of time samples $=257$, relative $L^{\infty}$ error $\approx 5$.E-4.


Phantom


Reconstruction Error(not to scale)

Relative error in $f(x)$ measured in $L^{2}(\Omega) \approx 0.6 \%$

## Next simulation, circular geometry with $50 \%$ noise (in $L^{2}$ )



Noisy data $g(t, \omega(\theta+\pi))$


Reconstruction from noisy data


Noisy data $g(t, \omega(0))$ vs exact Reconstructed $\operatorname{Rf}(\tau, \omega(0))$ vs exact Relative $L^{2}$ error in the reconstructed $\operatorname{Rf}(\omega, \tau)$ is $\approx 7 \%$.

## Reconstructing $f(x)$ from data with $50 \%$ noise



Phantom


Reconstruction Error(not to scale)

Relative error in $f(x)$ measured in $L^{2}(\Omega) \approx 28 \%$

## Spherical geometry

Polar coord: $y=(R, \psi)$, and $\omega=(\cos \varpi, \sin \varpi)$
Define $\varphi_{\omega}$ through its Fourier transform $\hat{\varphi}_{\omega}$ :

$$
\hat{\varphi}_{\omega}(\rho, \hat{y}(\psi))= \begin{cases}\frac{4 \pi}{i \rho} \sum_{k=0}^{\infty} \sum_{m=-k}^{k} \frac{i^{k} \overline{Y_{k}^{m}(\omega)} Y_{k}^{m}(\hat{y})}{h_{k}^{(1)}(\rho)}, & \rho \geq 0 \\ \frac{\hat{\varphi}_{\omega}(-\rho, \hat{y})}{}, & \rho<0\end{cases}
$$

## Truncated spherical geometry

Consider the following truncated spherical acquisition geometry:


Theorem: For the truncated spherical geometry, formula

$$
\mathcal{R} f(\omega, \tau)=\int_{0}^{\tau-T_{0}} \int_{\mathbb{S}^{2}} \widetilde{G(t, y)} \varphi_{\omega}(\tau-t, y) d y d t
$$

holds for all $\omega \neq(0,0,-1)$ and $\tau$ lying within the intervals

$$
\tau \in \begin{cases}\left(-1,-\cos \left(\frac{\pi}{4}-\nu\right)+\sin \frac{\pi}{4}\right), & \nu \in\left(0, \frac{\pi}{2}\right], \\ \left(-1,-\cos \left(\frac{\pi}{4}+\nu\right)-\sin \frac{\pi}{4}\right), & \nu \in\left[\frac{\pi}{2}, \pi\right] .\end{cases}
$$

## Simulation, spherical geometry, 3D



Our approach is quite general and is only explicit result for open and bounded acquisition surfaces.

We rely on the scattering problem by closed surfaces. For such surfaces there is a significant body of work on finding the density of single layer potentials and/or solving the scattering problem.

For certain surfaces reconstruction can be done analytically and results in fast algorithms.

Theoretically $\Omega^{-}$can have larger support, reconstruction is stable and unique (thanks to the visibility condition)

Thank you!

