Theoretically exact solution of the inverse source problem for the wave equation with spatially and temporally reduced data.

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> > August 08, 2019

The work was supported in part by NSF/DMS-1814592

TAT/PAT set up

Thermo-acoustic tomography



Short EM pulse is sent \Rightarrow EM energy is absorbed \Rightarrow Tissues heat and expand

 \Rightarrow Acoustic wave propagates

 \Rightarrow Acoustic pressure is measured

PAT: uses laser beams (TAT uses radio frequency pulses).

Inverse source problem of TAT/PAT: Find initial pressure from the measured pressure

TAT/PAT mathematical model

Acoustic pressure p(t, x) satisfies the wave equation Assume $c(x) \equiv 1$, no reflection, absorption, dispersion.

$$\begin{cases} p_{tt} = \Delta p, & x \in \mathbb{R}^n \\ p_t(0, x) = 0, & p(0, x) = f(x) \end{cases}$$



Measurements $g(t, y) \equiv p(t, y)$ done on S.

TAT/PAT inverse source problem reconstructs f(x) from g(t, y).

Solving the wave equation

Acoustic pressure p(t, x) satisfies the wave equation

$$\begin{cases} p_{tt} = \Delta p, & x \in \mathbb{R}^n \\ p_t(0, x) = 0, & p(0, x) = f(x) \end{cases}$$

Solution

$$p(t,y) \equiv \frac{\partial}{\partial t} \int_{\Omega^{-}} f(x) \Phi_n(t, x - y) dx, \text{ where}$$
$$\Phi_2(t,x) = \frac{H(t - |x|)}{2\pi\sqrt{t^2 - |x|^2}}, \Phi_3(t,x) = \frac{\delta(t - |x|)}{4\pi|x|}$$

are Green functions of the free-space wave equation. In particular, $g(t, y) \equiv p(t, y)$. *S* is a plane: multiple works "Universal formula" in 3D: a sphere, a plane, a cylinder (Xu & Wang)

Spheres (Finch et al; Kunyansky; Nguyen)

Ellipsoids and paraboloids (Natterer; Haltmeier; Palamodov;Salman) Limiting cases of ellipsoids and paraboloids (Haltmeier & Pereverzyev Jr.)

More complicated curves and surfaces (Palamodov)

Triangles, squares, cubes, and some tetrahedra (Kunyansky) Corner-like domains in 3D, a segment of Coxeter cross in 2D (Kunyansky)

Less explicit: series techniques (Kunyansky; Haltmeier et al)

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Less explicit: series techniques (Kunyansky; Haltmeier et al) All of these works either requires closed or unbounded *S*.

Why reduced data?

- Limited data in space: practical reasons (bounded observation surface, detectors not surrounded from all sides)
- Truncated data in time: increase accuracy (reflection, scattering)

Simply applying known formulas on truncated data? Artifacts!

Approximate and iterative techniques exist.

To obtain explicit, theoretically exact inversion formulas that use temporally truncated data measured on open surface S.

Our tools and approach

Radon projection of f(x)

$$\mathcal{R}f(\omega,\tau) = \int_{\omega\cdot x=\tau} f(x)dx$$

$$\mathcal{R}f(\omega,\tau) = \mathcal{R}f(-\omega,-\tau)$$



Our tools and approach

Radon projection of f(x)

$$\mathcal{R}f(\omega,\tau) = \int_{\omega \cdot x = \tau} f(x) dx$$



$$\mathcal{R}f(\omega, au) = \mathcal{R}f(-\omega,- au)$$

A filtered backprojection inversion formula

$$f(x) = \frac{1}{4\pi} \mathcal{R}^* \mathcal{H} \frac{\partial}{\partial t} \mathcal{R} f,$$

where

$$(\mathcal{R}^*h)(x) \equiv \int_{\mathbb{S}^1} h(x \cdot \omega, \omega) d\omega - \text{backprojection},$$

$$(\mathcal{H}u)(p) \equiv \text{p.v.} \frac{1}{\pi} \int_{\mathbb{R}} \frac{u(s)}{p-s} ds, \mathcal{H} \frac{\partial}{\partial t} - \text{filtration}.$$

Our tools and approach

Instead of f(x) we reconstruct its Radon transform $\mathcal{R}f$

$$\mathcal{R}f(\omega,\tau) = \int_{\omega\cdot x=\tau} f(x)dx = \int_{\Omega^-} f(x)\delta(-\omega\cdot x+\tau)dx$$

We want to represent the plane wave δ as a retarded single layer potential.

$$\delta(-\omega \cdot x + \tau) = \int_{T_0(\omega)}^{\tau} \int_{\Gamma} \Phi_n(\tau - t, x - y)\varphi_\omega(t, y) dy dt$$

using scattering theory!

Layer potentials and scattering theory

Consider an incoming wave $u^{inc} = u^-$. There will be a unique density $\varphi(t, y)$ defined on $\mathbb{R} \times \Gamma$ such that

$$u^{\pm}(\tau,x) = \int_{T_{0}(\omega)}^{\tau} \int_{\Gamma} \Phi_{n}(\tau-t,x-y)\varphi(t,y)dydt, x \in \Omega^{\pm},$$

where u^{\pm} solves the wave equation and u^{+} solves the soft scattering problem, i.e. satisfies the jump conditions on Γ $u^{+}(t, y) = u^{-}(t, y),$

$$\frac{\partial u^{-}(t,y)}{\partial n} - \frac{\partial u^{+}(t,y)}{\partial n} = \varphi(t,y).$$

Scattering problem for plane wave

Now for $u^{inc} = \delta(\tau - \omega \cdot x)$ one obtains

$$\delta(\tau - \omega \cdot x) = \int_{T_0(\omega)}^{\tau} \int_{\Gamma} \Phi_n(\tau - t, x - y) \varphi_\omega(t, y) dy dt,$$

Plane wave δ is causal: $u^{inc}(\tau, x) = 0$ if $\tau < \omega \cdot x$.

Due to the finite speed of propagation, both $u^- = u^{inc}$, and $u^+ = u^{scat}$ are 0 in front of the line $\omega \cdot x = t$.

Therefore, $\varphi_{\omega}(t, y)$ is also causal.

The sparse support of $\varphi_{\omega}(t, y)$ is crucial.



Measurements $g(t, y) \equiv p(t, y)$ on S are given by:

$$g(t,y) = \frac{\partial}{\partial t}G(t,y), G(t,y) \equiv \int_{\Omega^-} f(x)\Phi_n(t,x-y)dx, y \in S \subset \Gamma.$$

We want to recover the Radon projections of f(x) defined as

$$\mathcal{R}f(\omega, au)\equiv\int\limits_{\Omega^{-}}f(x)\delta(au-\omega\cdot x)dx.$$

$$\mathcal{R}f(\omega,\tau) = \int_{\Omega^{-}} f(x)\delta(-\omega \cdot x + \tau)dx$$

= $\int_{\Omega^{-}} f(x) \left[\int_{T_{0}}^{\tau} \int_{\Gamma} \Phi_{n}(\tau - s, x - y)\varphi_{\omega}(s, y)dyds \right] dx$
= $\int_{0}^{\tau-T_{0}} \int_{\Gamma} \left[\int_{\Omega^{-}} f(x)\Phi_{n}(t, x - y)dx \right] \varphi_{\omega}(\tau - t, y)dydt$
= $\int_{0}^{\tau-T_{0}} \int_{\Gamma} G(t, y)\varphi_{\omega}(\tau - t, y)dydt$

Do this for all $\omega \in \mathbb{S}^{n-1}$, $\tau \in \mathcal{T}(\omega)$, obtain $\mathcal{R}f(\omega, \tau)$.

Spatially limited data

$$\mathcal{R}f(\omega,\tau) = \int_{\Omega^{-}} f(x)\delta(-\omega \cdot x + \tau)dx$$
$$= \int_{\Omega^{-}} f(x) \left[\int_{T_{0}}^{\tau} \int_{\Gamma} \Phi_{n}(\tau - s, x - y)\varphi_{\omega}(s, y)dyds \right] dx$$



$$\operatorname{supp}(\operatorname{Er}(\delta(\tau,\omega)) = \bigcup_{y \in \Gamma \setminus S} \operatorname{B}(y,\tau-\omega \cdot y)$$

Inversion formula

Depending on ω , there is interval of values τ , for which equation

$$\mathcal{R}f(\omega, au) = \int_{0}^{ au-T_0} \int_{S} G(t,y) arphi_\omega(au-t,y) dy dt$$

is exact.

Theorem: For the following truncated acquisition geometry



we can reconstruct all Radon projections explicitly and exactly using temporally truncated data measured on an open surface S.

Circular and spherical acquisition surfaces

Note that

$$e^{i
ho\omega\cdot x} = \int\limits_{\mathbb{R}} \delta(t-\omega\cdot x)e^{i
ho t}dt$$

We represent $e^{i\rho\omega\cdot x}$ by a time harmonic single layer potential

$$e^{i
ho\omega\cdot x} = \int_{\mathbb{S}^{n-1}}\widehat{\varphi_{\omega}}(
ho,\hat{y})\hat{\Phi}_n(
ho,x-\hat{y})d\hat{y}$$

Take inverse Fourier transform, one obtain

$$\delta(t-\omega\cdot x)=\int_{\mathbb{S}^{n-1}}\int_{\mathbb{R}}arphi_{\omega}(t,\hat{y})\Phi_n(au-t,x-\hat{y})dtd\hat{y}$$

The densities are defined through their Fourier transforms.

Polar coord: $y = (R, \psi)$, and $\omega = (\cos \varpi, \sin \varpi)$

Define φ_{ω} through its Fourier transform $\hat{\varphi}_{\omega}$:

$$\hat{arphi}_{\omega}(
ho,\hat{y}(\psi)) = egin{cases} \sum\limits_{k=-\infty}^{\infty}rac{2j^{|k|}e^{ik(arphi-\psi)}}{\pi i \mathcal{H}^{(1)}_{|k|}(
ho)}, &
ho \geq 0 \ rac{\widehat{arphi}_{\omega}(-
ho,\hat{y}), &
ho < 0 \end{cases}$$

Truncated circular geometry

Consider the following truncated circular acquisition geometry:



Theorem: For the truncated circular geometry, formula

$$\mathcal{R}f(\omega,\tau) = \int_{0}^{\tau-T_0} \int_{\mathbb{S}^1} \widetilde{G(t,y)} \varphi_{\omega}(\tau-t,y) dy dt.$$

holds for all $\omega
eq (0,-1)$ and au lying within the intervals

$$\tau \in \left\{ \begin{array}{ll} \left(-1, -\cos\left(\frac{\pi}{4} - \nu\right) + \sin\frac{\pi}{4}\right), & \nu \in \left(0, \frac{\pi}{2}\right], \\ \left(-1, -\cos\left(\frac{\pi}{4} + \nu\right) - \sin\frac{\pi}{4}\right), & \nu \in \left[\frac{\pi}{2}, \pi\right]. \end{array} \right.$$

All Radon projections can be reconstructed exactly and explicitly from data measured on the open S acquiring in a reduced temporal range of $[0, 2 - 1/\sqrt{2}] \approx [0, 1.3]$ instead of the standard [0, 2].

Simulation, circular geometry, 2D

Our phantom is a collection of slightly smoothed characteristic functions of circles. *S* is the acquisition surface.



Solve wave equation find $g(t, \omega(\theta + \pi))$

Truncated $g(t, \omega(\theta + \pi))$



Reconstruction results, truncated circular geometry



Exact $Rf(\tau, \omega(\theta))$

Reconstruction w/o redundancy



Number of "detectors" = 512, reconstruction time = 0.4 sec.,

number of time samples= 257, relative L^{∞} error \approx 5.E-4.

Reconstructing f(x), truncated circular geometry



Phantom Reconstruction Error(not to scale)

Relative error in f(x) measured in $L^2(\Omega) \approx 0.6$ %

Next simulation, circular geometry with 50% noise (in L^2)



Noisy data $g(t, \omega(0))$ vs exact Reconstructed $Rf(\tau, \omega(0))$ vs exact Relative L^2 error in the reconstructed $Rf(\omega, \tau)$ is $\approx 7\%$.

Reconstructing f(x) from data with 50% noise



Phantom Reconstruction Error(not to scale)

Relative error in f(x) measured in $L^2(\Omega) \approx 28\%$

Polar coord: $y = (R, \psi)$, and $\omega = (\cos \varpi, \sin \varpi)$

Define φ_{ω} through its Fourier transform $\hat{\varphi}_{\omega}$:

$$\hat{arphi}_{\omega}(
ho,\hat{y}(\psi)) = egin{cases} rac{4\pi}{i
ho}\sum\limits_{k=0}^{\infty}\sum\limits_{m=-k}^{k}rac{i^k\overline{Y_k^m(\omega)}Y_k^m(\hat{y})}{h_k^{(1)}(
ho)}, &
ho \ge 0 \ rac{1}{\widehat{arphi}_{\omega}(-
ho,\hat{y})}, &
ho < 0 \end{cases}$$

Consider the following **truncated** spherical acquisition geometry:



Theorem: For the truncated spherical geometry, formula

$$\mathcal{R}f(\omega,\tau) = \int_{0}^{\tau-T_0} \int_{\mathbb{S}^2} \widetilde{G(t,y)} \varphi_{\omega}(\tau-t,y) dy dt.$$

holds for all $\omega
eq (0,0,-1)$ and au lying within the intervals

$$\tau \in \left\{ \begin{array}{ll} \left(-1, -\cos\left(\frac{\pi}{4} - \nu\right) + \sin\frac{\pi}{4}\right), & \nu \in \left(0, \frac{\pi}{2}\right], \\ \left(-1, -\cos\left(\frac{\pi}{4} + \nu\right) - \sin\frac{\pi}{4}\right), & \nu \in \left[\frac{\pi}{2}, \pi\right]. \end{array} \right.$$

Simulation, spherical geometry, 3D



Our approach is quite general and is only explicit result for open and bounded acquisition surfaces.

We rely on the scattering problem by closed surfaces. For such surfaces there is a significant body of work on finding the density of single layer potentials and/or solving the scattering problem.

For certain surfaces reconstruction can be done analytically and results in fast algorithms.

Theoretically Ω^- can have larger support, reconstruction is stable and unique (thanks to the visibility condition)

Thank you!