Learning the Invisible: Limited Angle Tomography, Shearlets and Deep Learning

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Modern Challenges in Imaging
In the Footsteps of Allan MacLeod Cormack
On the Fortieth Anniversary of his Nobel Prize

Tufts University, 5–9 August 2019

Finnish Inverse Problems Society

UNIVERSITY OF HELSINKI
FACULTY OF SCIENCE

Finnish Centre of Excellence in Inverse Modelling and Imaging
2018-2025

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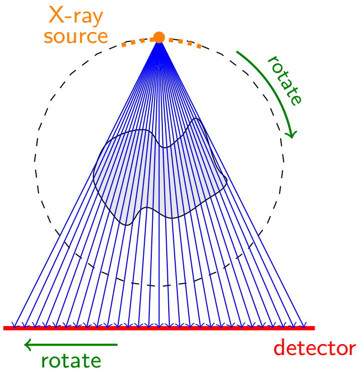
Computed Tomography



Mathematically, a CT scanner samples the **Radon transform**:

$$\mathcal{R}f(\theta, s) = \int_{L(\theta,s)} f(\boldsymbol{x}) dS(\boldsymbol{x}),$$

where $\theta \in [-\pi/2, \pi/2)$, $s \in \mathbb{R}$ and
$L(\theta, s) := \{\boldsymbol{x} \in \mathbb{R}^2 : x_1 \cos(\theta) + x_2 \sin(\theta) = s\}$.

Task: Recover the scanned object f from the given data $\mathcal{R}f(\theta, s)$.

Difficulties: Challenging inverse problem when θ and s scarcely sampled.

Linear Discrete Model

For $\boldsymbol{f}_{\mathtt{gt}} \in \mathbb{R}^{n^2}$ obtain the linear model

$$\boldsymbol{y} = \boldsymbol{\mathcal{R}}\,\boldsymbol{f}_{\mathtt{gt}} + \epsilon, \tag{1}$$

where $\boldsymbol{\mathcal{R}} \in \mathbb{R}^{m \times n^2}$ (*discretized line integrals*) and $\|\epsilon\|_2 \leqslant \eta$ *(noise)*.

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where $\boldsymbol{\mathcal{R}} \in \mathbb{R}^{m \times n^2}$ (*discretized line integrals*) and $\|\epsilon\|_2 \leqslant \eta$ (*noise*).

The resulting ill-posed/ill-conditioned inverse problem is usually solved using *filtered backprojection* (FBP), iterative reconstruction, or **regularization**:

$$\underset{\boldsymbol{f} \geqslant 0}{\operatorname{argmin}} \left\{ \frac{1}{2} \| \boldsymbol{\mathcal{R}}\, \boldsymbol{f} - \boldsymbol{y} \|_2^2 + R(\boldsymbol{f}) \right\}, \tag{2}$$

where R is chosen for instance as

- $R(\boldsymbol{f}) = \|\mathbf{L}\boldsymbol{f}\|_2^2$: (Generalized) Tikhonov,
- $R(\boldsymbol{f}) = \|\mathbf{L}\boldsymbol{f}\|_1$: "compressed sensing" or sparse regularization.

where $\mathbf{L}$ is a suitable operator.

Limited Angle Tomography

Sample $\mathcal{R}f(\cdot, s)$ on $[-\phi, \phi] \subset [-\pi/2, \pi/2)$, denoted by $\mathcal{R}_\phi f = \mathcal{R}f_{|[-\phi,\phi]\times\mathbb{R}}$.

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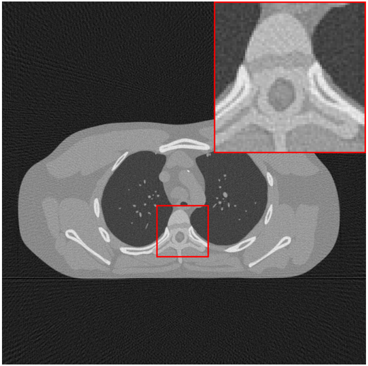

$\phi = 90^\circ$, filtered backprojection (FBP)

Limited Angle Tomography

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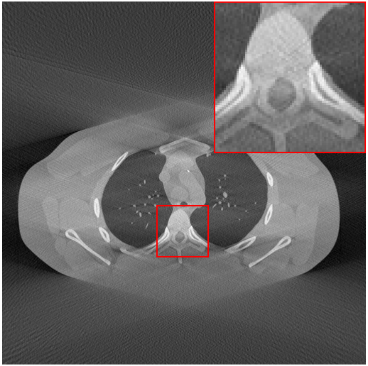
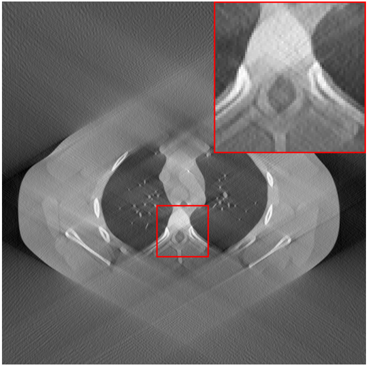


$\phi = 75^\circ$, filtered backprojection (FBP)

Limited Angle Tomography

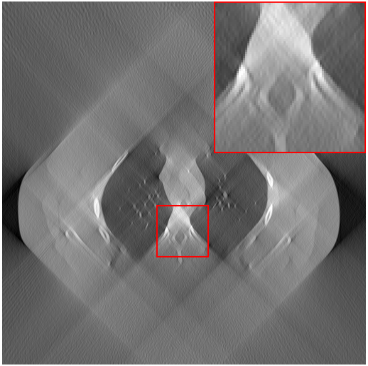
Sample $\mathcal{R}f(\cdot, s)$ on $[-\phi, \phi] \subset [-\pi/2, \pi/2)$, denoted by $\mathcal{R}_\phi f = \mathcal{R}f_{|[-\phi,\phi]\times\mathbb{R}}$.



$\phi = 60^\circ$, filtered backprojection (FBP)

Limited Angle Tomography

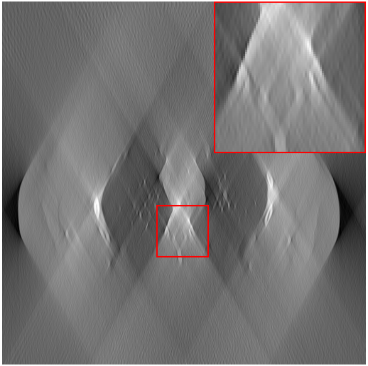
Sample $\mathcal{R}f(\cdot, s)$ on $[-\phi, \phi] \subset [-\pi/2, \pi/2)$, denoted by $\mathcal{R}_\phi f = \mathcal{R}f_{|[-\phi,\phi]\times\mathbb{R}}$.



$\phi = 45^\circ$, filtered backprojection (FBP)

Limited Angle Tomography

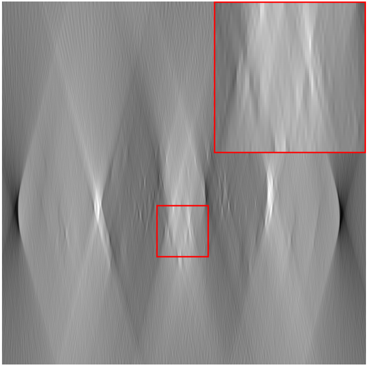
Sample $\mathcal{R}f(\cdot, s)$ on $[-\phi, \phi] \subset [-\pi/2, \pi/2)$, denoted by $\mathcal{R}_\phi f = \mathcal{R}f_{|[-\phi,\phi]\times\mathbb{R}}$.



$\phi = 30^\circ$, filtered backprojection (FBP)

Limited Angle Tomography

Sample $\mathcal{R}f(\cdot, s)$ on $[-\phi, \phi] \subset [-\pi/2, \pi/2)$, denoted by $\mathcal{R}_\phi f = \mathcal{R}f_{|[-\phi,\phi]\times\mathbb{R}}$.



$\phi = 15^\circ$, filtered backprojection (FBP)

Observations:

- only certain boundaries/features seem to be **"visible"**,
- missing wedge creates artifacts,
- highly ill-posed inverse problem!

Visibility in CT: Microlocal Analysis & Wavefront Sets

"visible": singularities tangent to sampled lines

"invisible": singularities not tangent to sampled lines

Main literature for limited data CT:

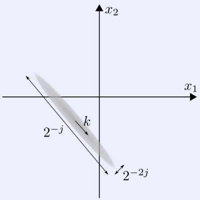
- characterization in sinogram: **Quinto (1993)**
- characterization in FBP, reduction of artifacts: **Frikel & Quinto (2013)**

Why Shearlets?

Definition (Discrete shearlet system [Kutyniok & Labate, 2006])

Let $\psi \in L^2(\mathbb{R}^2)$. For $j \in \mathbb{Z}$ and $k \in \mathbb{R}$, define the parabolic scaling matrix A_j and the shear matrix S_k by

$$A_j := \begin{bmatrix} 2^{2j} & 0 \\ 0 & 2^j \end{bmatrix}, \quad S_k := \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$



The **discrete shearlet system** defined by ψ is:

$$\mathcal{SH}(\psi) := \{\psi_{j,k,m} := 2^{3j/2}\psi(S_k A_j \cdot - m) \mid j \in \mathbb{Z}, k \in \mathbb{Z}, m \in \mathbb{Z}^2\}.$$

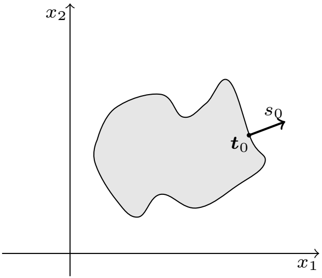
- Analogous continuous setting with parameters $a \in \mathbb{R}_+$, $s \in \mathbb{R}$, $\boldsymbol{t} \in \mathbb{R}^2$.
- Many more evolved constructions (cone-adapted, bandlimited, compactly supported, ...).

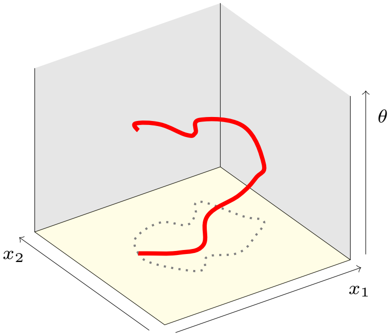
G. Kutyniok and D. Labate, *Shearlets: Multiscale Analysis for Multivariate Data*, 1st ed. New York: Springer Verlag, 2012.

Shearlets and Wavefront Sets

By using continuous shearlets theory:

- f is *smooth* in $\boldsymbol{t}_0$ and shearing direction s_0
  $\Longrightarrow$ fast decay of shearlet coefficients $\Longrightarrow$ **sparsity!**
- wavelets only characterize *singular support* of f (*i.e.*, no directional information)





$*$-lets and Limited Angle Tomography I

- Forward problem carefully analyzed by Frikel (2013) for curvelets

The index set of curvelets can be split into a part that is "visible" under $\mathcal{R}_\phi$ and a part that is "invisible", *i.e.*, $\mathcal{R}_\phi \, \psi_{j,k,\iota} = 0$ (via *Fourier slice theorem*).

J. Frikel, *Sparse regularization in limited angle tomography*, Appl. Comp. Harm. Anal. **34** (1), 117–141, 2013.

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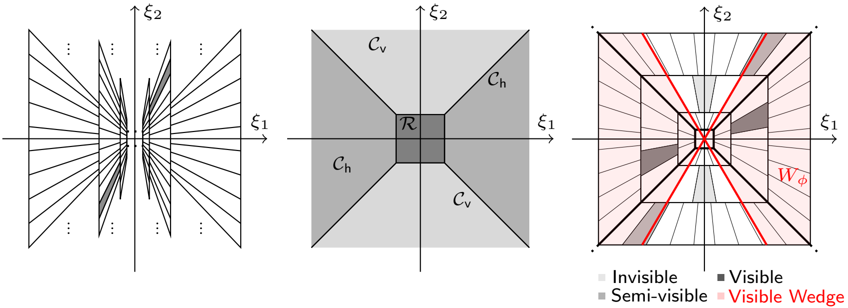
- Holds for shearlets and other directional representation systems as well



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$*$-lets and Limited Angle Tomography II

Obtain

$$
\begin{aligned}
f &= \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\mathtt{vis}}} \langle f, \psi_{j,k,\boldsymbol{l}}\rangle\, \psi_{j,k,\boldsymbol{l}} + \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\mathtt{inv}}} \langle f, \psi_{j,k,\boldsymbol{l}}\rangle\, \psi_{j,k,\boldsymbol{l}} \\
&= f_{\mathtt{vis}} + f_{\mathtt{inv}}.
\end{aligned}
$$

Use in:

$$
\mathrm{SH}_\psi^T \left(\operatorname*{argmin}_{\boldsymbol{z}} \|\boldsymbol{z}\|_{1,\boldsymbol{w}} + \frac{1}{2} \| \boldsymbol{\mathcal{R}}_{\boldsymbol{\phi}}\, \mathrm{SH}_\psi^T(\boldsymbol{z}) - \boldsymbol{y}\|_2^2 \right)
$$

$*$-lets and Limited Angle Tomography II

Obtain

$$f = \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\mathtt{vis}}} \langle f, \psi_{j,k,\boldsymbol{l}} \rangle\, \psi_{j,k,\boldsymbol{l}} + \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\mathtt{inv}}} \langle f, \psi_{j,k,\boldsymbol{l}} \rangle\, \psi_{j,k,\boldsymbol{l}}$$

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- dimension reduction

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Obtain

$$\begin{aligned} f &= \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\texttt{vis}}} \langle f, \psi_{j,k,\boldsymbol{l}}\rangle\, \psi_{j,k,\boldsymbol{l}} + \sum_{(j,k,\boldsymbol{l})\in\mathcal{I}_{\texttt{inv}}} \langle f, \psi_{j,k,\boldsymbol{l}}\rangle\, \psi_{j,k,\boldsymbol{l}} \\ &= f_{\texttt{vis}} + f_{\texttt{inv}}. \end{aligned}$$

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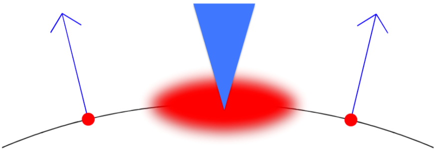
- dimension reduction
- drawbacks: no positivity constraint, synthesis formulation

The Idea

Facts:

- parts of the WF are available only "here and there".
- shearlets are proven to resolve the WF



Idea: use shearlet coefficients to fill in the missing parts of the wavefront set.

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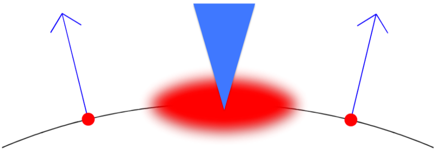

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ℓ^1-minimization reconstruction

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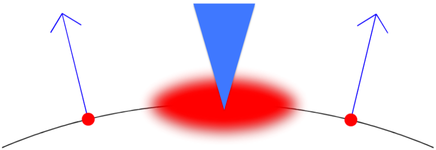

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ℓ^1-minimization reconstruction $\longrightarrow$ shearlet cube: visibile + invisible

"candy-wrap"

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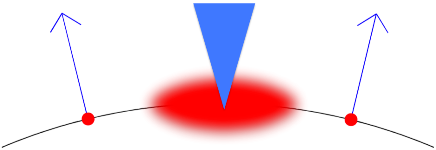

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ℓ^1-minimization reconstruction $\longrightarrow$ shearlet cube: visibile + invisible $\longrightarrow$ "inpainted" reconstruction

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fill in missing coeff.

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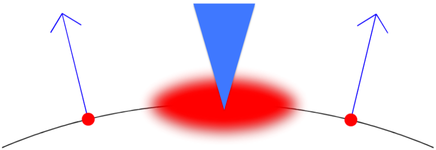

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ℓ^1-minimization reconstruction $\longrightarrow$ shearlet cube: visibile + invisible $\xrightarrow{\text{DL?}}$ "inpainted" reconstruction

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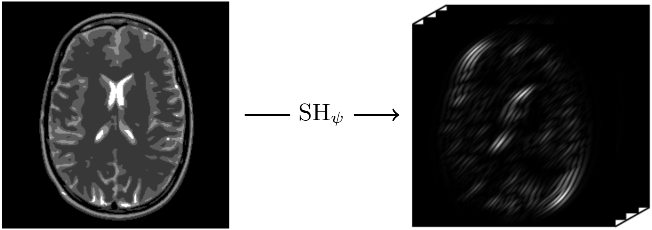
fill in missing coeff.

Shearlet Cube

Numerically, the shearlet transform does the following:

$$\boldsymbol{f} \in \mathbb{R}^{n\times n} \longmapsto \boldsymbol{F} \in \mathbb{R}^{n\times n\times L}$$

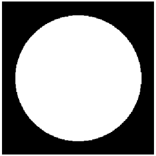


$\boldsymbol{f} \in \mathbb{R}^{n\times n}$ $\qquad$ $\boldsymbol{F} \in \mathbb{R}^{n\times n\times L}$

- Each subband $\boldsymbol{F}(:,:,i)$ corresponds to the inner products $\langle \boldsymbol{f}, \psi_{j,k,\cdot}\rangle$,
- $\boldsymbol{F}$ is referred to as the *shearlet coefficients cube* of $\boldsymbol{f}$.

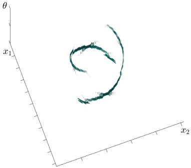
"Candy–wrap" Structure

Due to the sorting, shearlets coefficients follow a specific structure in each scale:



$\boldsymbol{f} \in \mathbb{R}^{n \times n}$

$\text{—}\ \mathrm{SH}_\psi \rightarrow$



highest scale

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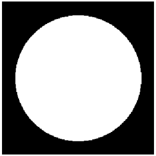
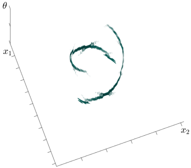


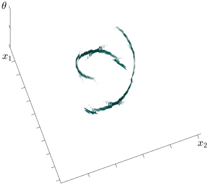
$\boldsymbol{f} \in \mathbb{R}^{n \times n}$

$\longrightarrow \mathrm{SH}_\psi \rightarrow$

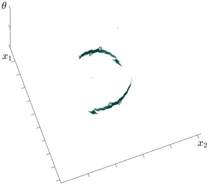


highest scale

Invisibility of limited angle tomography "creates" holes in it:



(a) full-angle CT



(b) limited angle CT

“Candy–wrap” Prior?

First Idea

Handcraft a prior that promotes this specific “candy–wrap” structure of the shearlet coefficients: enforce continuity of WF.

Too complicated?

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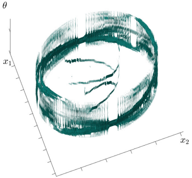
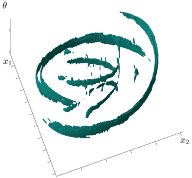
- "messy" shearlet coefficients
- easy rule for the human eye, hard to grasp mathematically



(a) shearlet cube for Shepp-Logan



(b) "cleaned" shearlet cube

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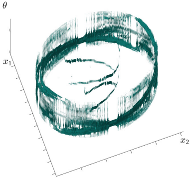

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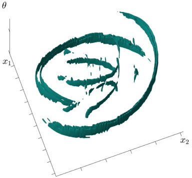


(b) "cleaned" shearlet cube

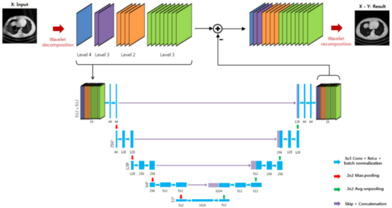
Idea

Train a deep neural network to fill in the gaps of the WF.

Some Literature on DL for Inverse Problems

- on FBP:
  - Kang et al. (2017): contourlets of FBP + U-net, 2nd place Mayo low-dose challenge & many more works from this group!
  - Zhang et al. (2016): 2-layer network on FBP
  - Jin et al. (2017): U-Net on FBP
- incorporating forward model via optimization scheme:
  - Hammernik et al. (2017): learning weights for FBP, then filtering
  - Meinhardt et al. (2017): learning proximal operators
  - Adler et al. (2017): learned primal dual

The Closest Method: Gu & Ye (2017)



"Based on the observation that the **artifacts from limited angles have some directional property** and are globally distributed, we propose a novel multi-scale wavelet domain residual learning architecture, which compensates for the artifacts."

Some Observations

Concerning "denoising" of the FBP (or its coefficients) with DL:

- missing theory, unclear what the NN really does:
  - entire image is processed
  - which features are modified?
  - lack of a clear interpretation (?)
- NN needs to learn a lot of streaking artifacts (+ noise)

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We can do better (in limited angle CT)!

- only the invisible boundaries need to be learned
- shearlets help to access them
- the coefficients follow the "candy–wrap" structure

Our Approach

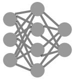
Step 1, "recover the visible": best available classical solution (little artifacts, denoised)

$$\boldsymbol{f}^* := \underset{\boldsymbol{f} \geqslant 0}{\operatorname{argmin}} \, \| \, \mathrm{SH}_\psi(\boldsymbol{f}) \|_{1,w} + \frac{1}{2} \| \, \boldsymbol{\mathcal{R}}_{\boldsymbol{\phi}} \, \boldsymbol{f} - \boldsymbol{y} \|_2^2$$

Allows to access WF via sparsity prior on shearlets:

- for $(j, k, \boldsymbol{l}) \in \mathcal{I}_{\mathtt{inv}}$: $\mathrm{SH}_\psi(\boldsymbol{f}^*)_{(j,k,\boldsymbol{l})} \approx 0$
- for $(j, k, \boldsymbol{l}) \in \mathcal{I}_{\mathtt{vis}}$: $\mathrm{SH}_\psi(\boldsymbol{f}^*)_{(j,k,\boldsymbol{l})}$ reliable and near perfect

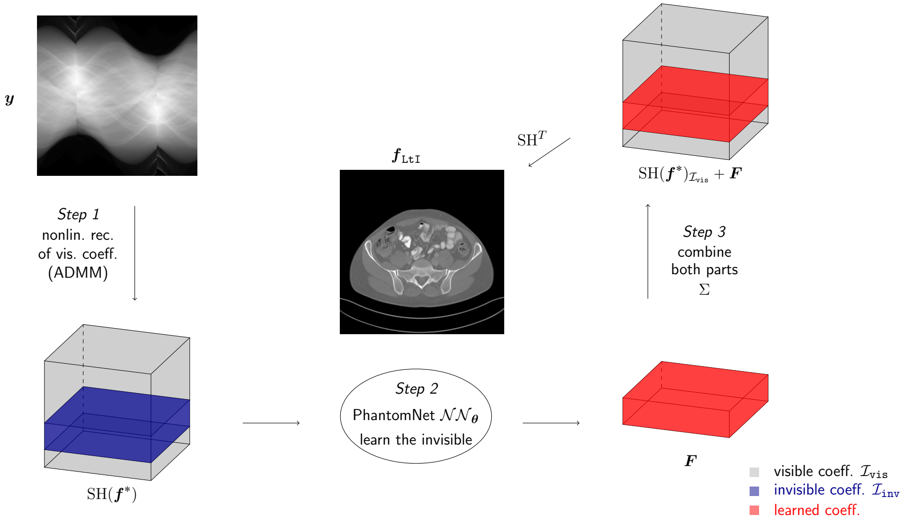
Step 2, "learn the invisible": supervised learning of invisible coefficients

$\mathcal{N}\mathcal{N}_{\boldsymbol{\theta}} : \; \mathrm{SH}_\psi(\boldsymbol{f}^*)_{\mathcal{I}_{\mathtt{inv}}}$ ——  $\longrightarrow$ $\boldsymbol{F} \; \left(\overset{!}{\approx} \mathrm{SH}_\psi(\boldsymbol{f}_{\mathtt{gt}})_{\mathcal{I}_{\mathtt{inv}}} \right)$

Step 3, "combine":

$$\boldsymbol{f}_{\mathtt{LtI}} = \mathrm{SH}_\psi^T \left(\mathrm{SH}_\psi(\boldsymbol{f}^*)_{\mathcal{I}_{\mathtt{vis}}} + \boldsymbol{F} \right)$$

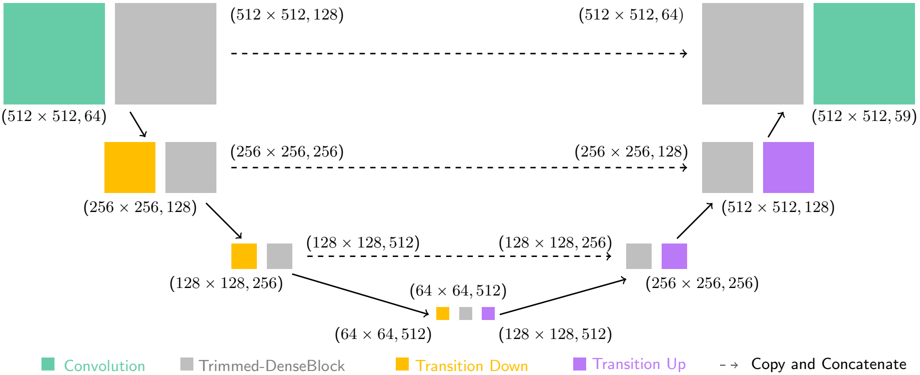
Our Approach



Our Approach – Step 2: CNN PhantomNet

Convolutional Neural Network that minimizes over the empirical risk:

$$\min_{\boldsymbol{\theta}} \frac{1}{N} \sum_{j=1}^{N} \| \mathcal{N}\mathcal{N}_{\boldsymbol{\theta}}(\mathrm{SH}(\boldsymbol{f}_j^*)) - \mathrm{SH}(\boldsymbol{f}_j)_{\mathcal{I}_{\mathtt{inv}}} \|_{\boldsymbol{w},2}^2 .$$



Learning the Invisible

Model based & data driven: only learn what needs to be learned!

Possible advantages:

- faithfulness by learning only what is not visible in the data
- better performance due to better input
- NN does not process entire image
  - less blurring by U-net
  - fewer unwanted artifacts
- better generalization

T.A. Bubba, G. Kutyniok, M. Lassas, M. März, W. Samek, S. Siltanen and V. Srinivasan, *Learning the Invisible: a hybrid deep learning-shearlets framework for limited angle computed tomography*, Inverse Problems **35**, 064002.

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Disadvantage:

- speed: dominated by ℓ^1-minimization

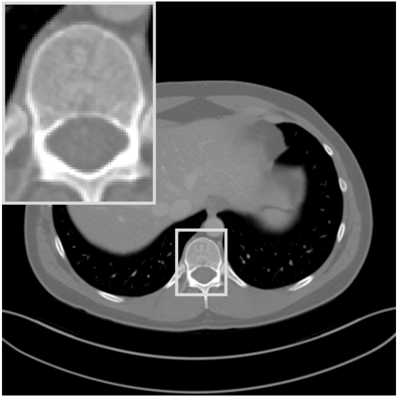
T.A. Bubba, G. Kutyniok, M. Lassas, M. März, W. Samek, S. Siltanen and V. Srinivasan, *Learning the Invisible: a hybrid deep learning-shearlets framework for limited angle computed tomography*, Inverse Problems **35**, 064002.

Setup

- Experimental Scenarios:
  - Mayo Clinic[4]: human abdomen scans provided by the Mayo Clinic for the AAPM Low-Dose CT Grand Challenge.
    - 10 patients (2378 slices of size 512×512 with thickness 3mm)
    - 9 patients for training (2134 slices) and 1 patient for testing (244 slices)
    - Mayo-60°: missing wedge of 60°
    - Mayo-75°: missing wedge of 30°
  - Lotus Root: real data measured with the μCT in Helsinki
    - to check generalization properties of our method (training is on Mayo-60°)
    - Lotus-60°: missing wedge of 60°
    - Lotus-75°: missing wedge of 30°
- Operators:
  - $\boldsymbol{\mathcal{R}}_{\boldsymbol{\phi}}$: Astra toolbox (fanbeam geometry fits lotus root's acquisition setup)
  - SH_ψ: α-shearlet transform toolbox (bandlimited shearlets with 5 scales, *i.e.*, 59 subbands of size 512×512)

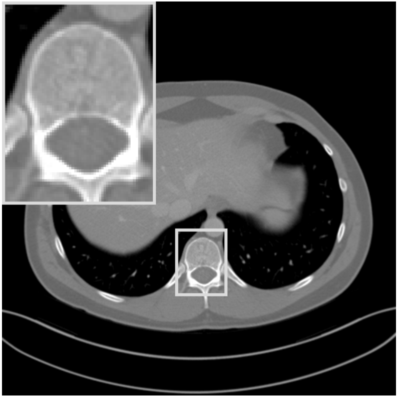
[4] We would like to thank Dr. Cynthia McCollough, the Mayo Clinic, the American Association of Physicists in Medicine (AAPM), and grant EB01705 and EB01785 from the National Institute of Biomedical Imaging and Bioengineering for providing the Low-Dose CT Grand Challenge data set.

Mayo-60°

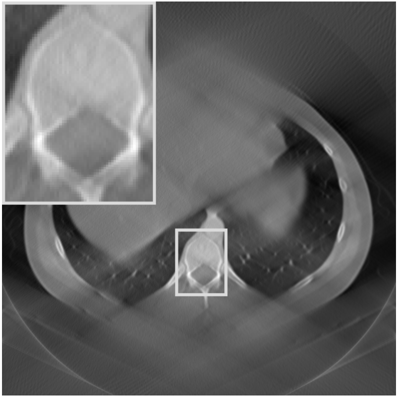


$f_{\mathtt{gt}}$

Mayo-60°

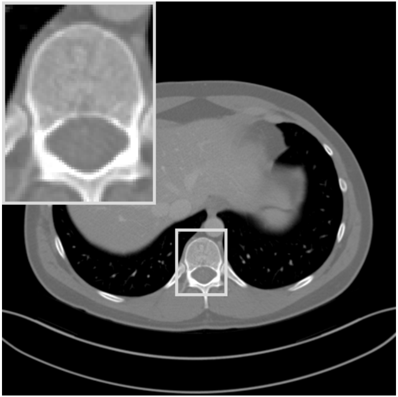


f_{gt}

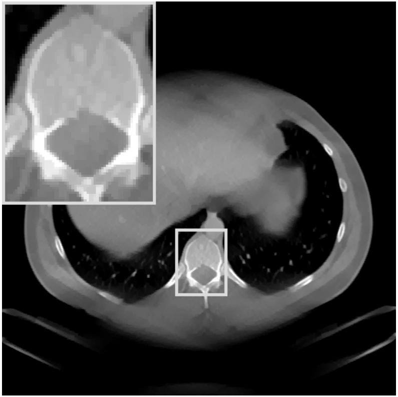


f_{FBP}: RE $= 0.50$, HaarPSI$=0.35$

Mayo-60°

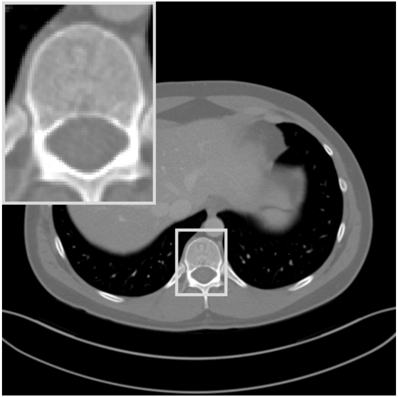


$\boldsymbol{f}_{\mathtt{gt}}$

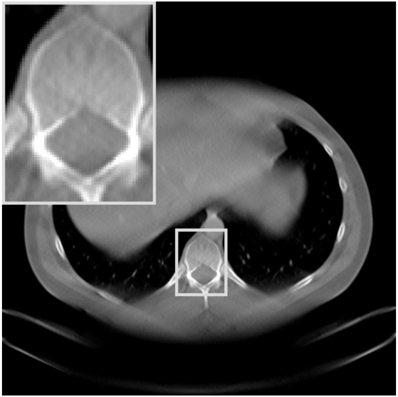


$\boldsymbol{f}_{\mathtt{TV}}$: RE = 0.21, HaarPSI=0.41

Mayo-60°

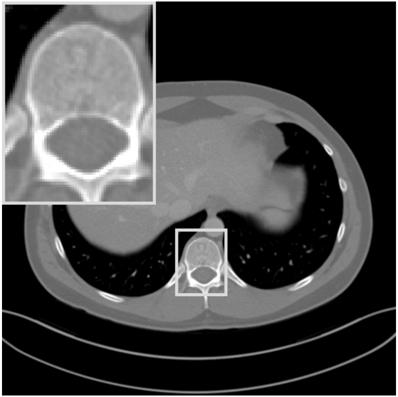


f_{gt}



f^*: RE = 0.19, HaarPSI=0.43

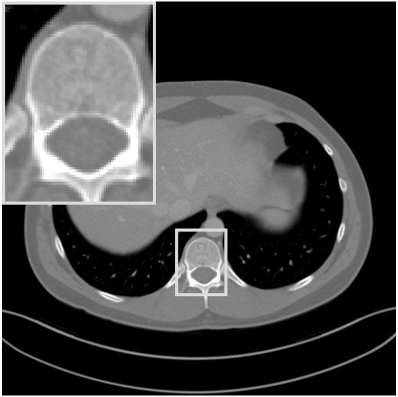
Mayo-60°



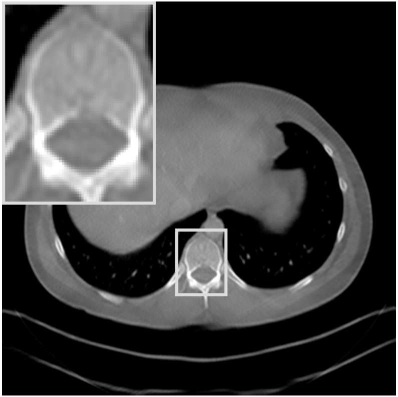
$\boldsymbol{f}_{\mathrm{gt}}$

$\boldsymbol{f}_{[\text{Gu \& Ye, 2017}]}$: RE = 0.22, HaarPSI=0.40

Mayo-60°

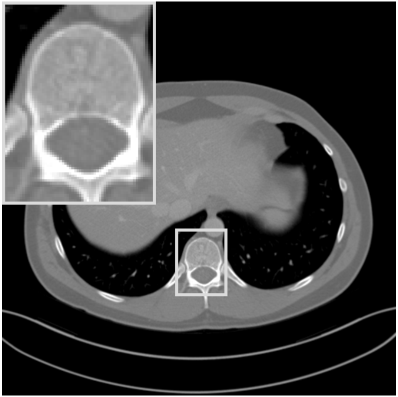


$\boldsymbol{f}_{\mathrm{gt}}$



$\mathcal{NN}_{\boldsymbol{\theta}}(\boldsymbol{f}_{\mathrm{FBP}})$: RE $= 0.16$, HaarPSI$=0.53$

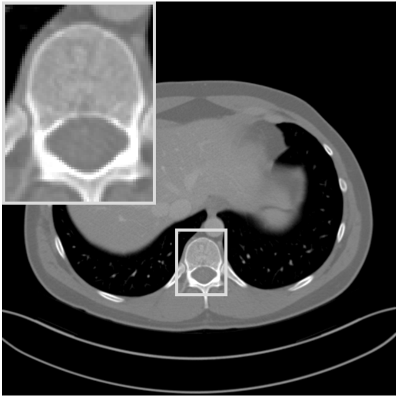
Mayo-60°



$\boldsymbol{f}_{\mathrm{gt}}$

$\mathcal{NN}_{\boldsymbol{\theta}}(\mathrm{SH}(\boldsymbol{f}_{\mathrm{FBP}}))$: RE = 0.16, HaarPSI=0.58

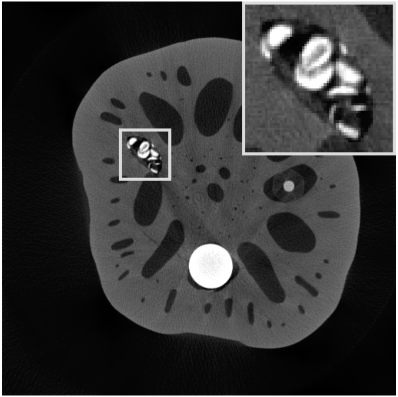
Mayo-60°



f_{gt}

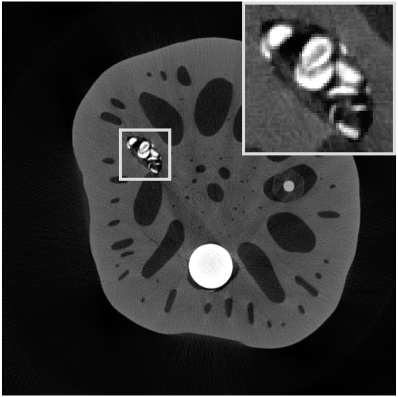
f_{LtI}: RE $= 0.09$, HaarPSI$=0.76$

Lotus-60°

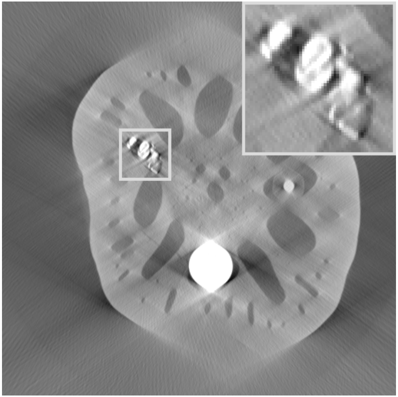


f_{gt}

Lotus-60°

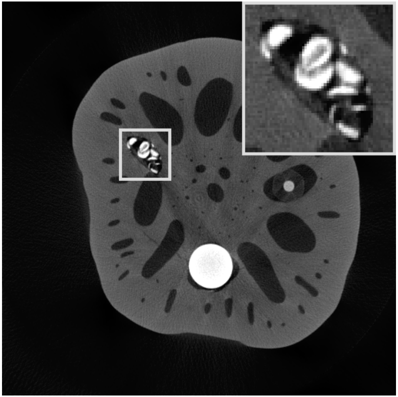


f_{gt}

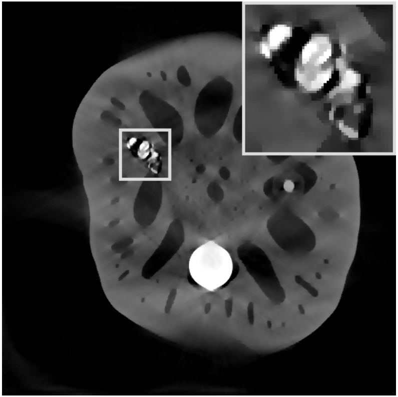


f_{FBP}: RE = 0.49, HaarPSI=0.49

Lotus-60°

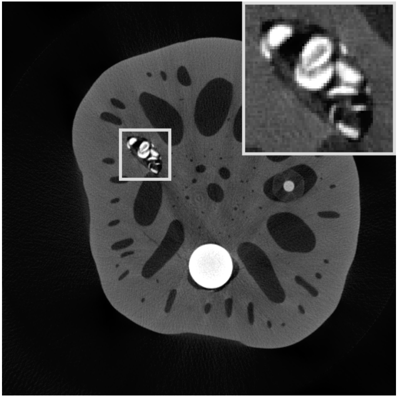


f_{gt}

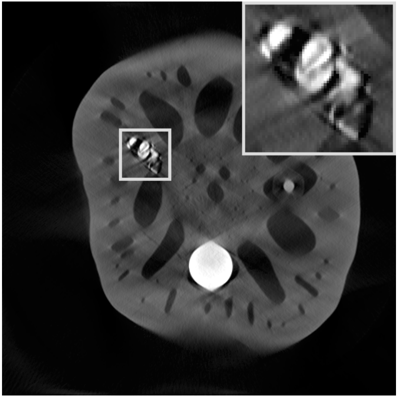


f_{TV}: RE $= 0.21$, HaarPSI$=0.60$

Lotus-60°

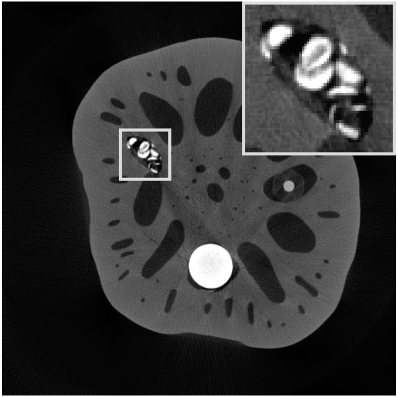


f_{gt}

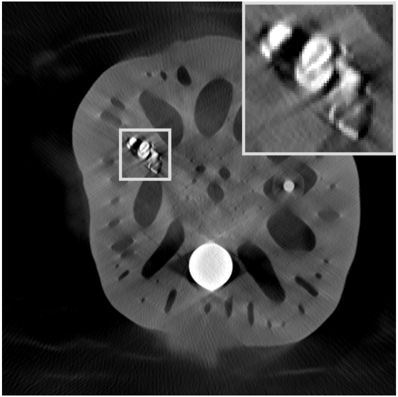


f^*: RE = 0.19, HaarPSI=0.61

Lotus-60°

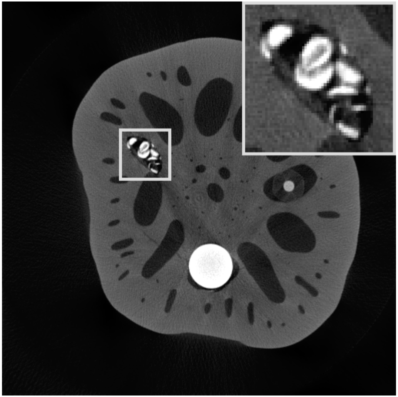


f_{gt}

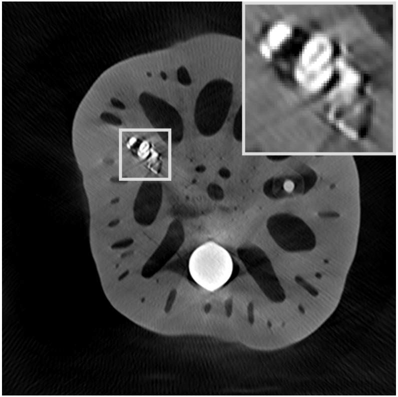


$f_{\text{[Gu \& Ye, 2017]}}$: RE $= 0.42$, HaarPSI$=0.56$

Lotus-60°

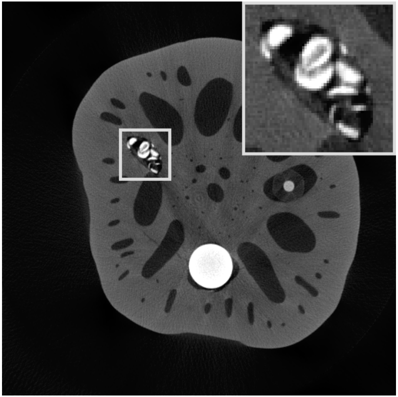


$\boldsymbol{f}_{\mathrm{gt}}$

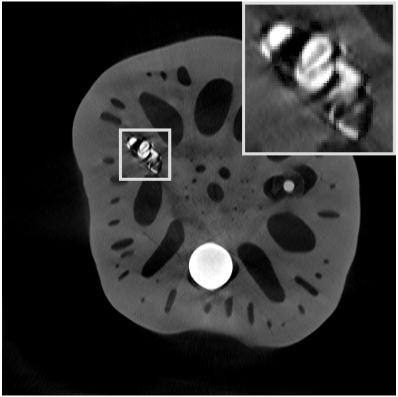


$\mathcal{NN}_{\boldsymbol{\theta}}(\mathrm{SH}(\boldsymbol{f}_{\mathrm{FBP}}))$: RE $= 0.27$, HaarPSI$=0.63$

Lotus-60°



f_{gt}



$f_{\mathtt{LtI}}$: RE $= 0.15$, HaarPSI$=0.74$

Conclusions

- limited angle CT is a special inverse problem
  - visible and invisible features
- ℓ^1-minimization with shearlets
  - access visible part of WF
  - negligible invisible part
- learn the invisible parts with a deep NN
  - 3D "inpainting" problem
  - regularity assumptions on f
- faithful approach: limit influence of DL
  - no explanation of DL
  - but *clearer* concept what is happening!

Future Perspectives

- Consider smaller visible wedges (*e.g.*, in breast CT $\phi = 20^\circ$ with 11 sampled angles)

- Optimize over a more sophisticated loss function:
  - by adding additional regularizers
  - by defining the loss function over the image domain

- Apply the same machinery to other limited or scarce data CT problems:
  - region of interest or exterior tomography
  - real life applications where most of the data are not acquired in the measurements


Thank you!