Single and paired shifted Funk transforms

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The Funk-Minkowski-Radon transform

Funk transform evaluates the integrals over the great $(n-2)-$ dim spheres in S^{n-1} :

$$F : C(S^{n-1}) \to C(S^{n-1}),$$

$$Ff(\omega) = \int\limits_{x \in S^{n-1} \cap \{\langle x, \omega \rangle = 0\}} f(x)\, dA^{n-1}(x),$$

where $\omega \in S^{n-1}$, and dA^{n-2} is Lebesgue surface area measure.

The classical Funk transform

Paul Funk (1911). Applications: diffusion MRI ...

- $\ker F = C_-(S^{n-1})$- the subspace of odd functions.
- The restriction on even functions:

$$F_+ : C_+(S^{n-1}) \to C_+(S^{n-1})$$

  is surjective and injective.
- Helgason's inversion formula ($n = 3$):

$$(F_+^{-1}g)(x) = \frac{1}{2\pi} \left[\frac{d}{ds} \int\limits_0^\infty (F^*g)(\arccos\ v, x) v(s^2 - v^2)^{-\frac{1}{2}} dv \right] |_{s=1},$$

$$(F^*g)(p, x) = \frac{1}{2\pi cosp} \int\limits_{|u|=1, \langle x,u \rangle = sin\ p} g(u)\ du.$$

- F_+^{-1} reconstructs the even parts: $f_+ = F_+^{-1}Ff$ for any $f \in C(S^{n-1})$.

Shifted Funk transform

Integration over non-central cross-sections

Definition

Let $a \in \mathbb{R}^n$. The (shifted) Funk transform with the center a is defined on $f \in C(S^{n-1})$ by

$$(F_a f)(\omega) = \int_{S^{n-1} \cap \{\langle x-a, \omega\rangle = 0\}} f(x)\, dA(x).$$

Y. Salman (2016, 2017).

The classical Funk transform is F_0, $(a = 0)$.

Shifted Funk transform, interior center

Questions: kernel, range, inversion

- The case $|a| = 1$: A. Aboulaz and R. Daher (1993), S. Helgason (2011), B. Rubin (2105).

- $|a| < 1$. Description of *ker* F_a. Link between the transforms F_a and F. Inversion of F_a. M. Quellmalz (2017) ($n = 3$); M. Quellmalz, B. Rubin (2018) ($n \geq 3$).

Formulation of the problems:

- Single SFT: The case of arbitrary location of the center; kernel, reconstruction?

- Paired SFT. Whether / when two Funk data (F_a, F_b) are enough for the reconstruction of functions?
- Define the paired Funk transform

  $(F_a, F_b) : C(S^{n-1}) \to C(S^{n-1}) \times C(S^{n-1}),\ (F_a, f_b)f = (F_a f, F_b f).$

- Describe

$$Ker(F_a, F_b) = Ker\ F_a \cap Ker\ F_b.$$

- For what pairs (a, b) the transform (F_a, F_b) is injective? $Ker\ F_a \cap Ker\ F_b = \{0\}$?

- In the case (F_a, F_b) is injective, recover f from $(F_a f, F_b f)$.

Standard transforms

- The center at 0: the Funk transform $F = F_0$.
- The center at ∞. The parallel slice Funk transform :

$$V_a f(\omega, t) = \int\limits_{\langle x, \omega \rangle = t} f(x)\, dV_{n-1}(x),$$

  where $\omega \perp a$. V_a integrates functions over the cross-sections of S^{n-1}, parallel to a.
- $V_a f = \lim\limits_{\lambda \to +\infty} F_{\lambda a} = F_{\infty \cdot a}$.
- $Ker\ V_a = \{f : f(x) = -f(\sigma_b x)\},\ \sigma_b x = x - 2\frac{\langle x, b \rangle}{|b|^2} b$.
- V_a - is inverted on b- even functions, R. Hielscher and M. Quellmalz, (2016) ($n = 3$,) B. Rubin (2018).

Intertwining between SFT and FT

Theorem

$$F_a f(\omega) = F_0\Big((f \circ \varphi_a) J_a\Big)(\varphi_a^* \omega); \quad |a| < 1,$$

$$F_a f(\omega) = V_a\Big((f \circ \varphi_{a^*}) J_{a^*}\Big)(\varphi_{a^*}^* \omega); \quad |a| > 1,$$

$$\varphi_a x = \frac{P_a(a-x) + \sqrt{1-|a|^2}\, Q_a(a-x)}{1 - \langle x, a\rangle}, \quad \varphi_a^* \omega = \frac{P_a\omega + s_a Q_a \omega}{|P_a\omega + s_a Q_a\omega|}.$$

$P_a x = \frac{\langle x,a\rangle}{|a|^2} a$, $Q_a = I - P_a$*-orthogonal projections,*

$J_a(x) = \left(\frac{\sqrt{1-|a|^2}}{1-\langle x,a\rangle}\right)^{n-2}$, $a^* = \frac{a}{|a|^2}$.

Ingredients of the proof of the intertwining relations

- The main tool: the group $Aut(B^n)$ of hyperbolic automorphisms.

- Fractional-linear extension of $Aut(B^n)|_{S^{n-1}}$ into B^n. The extensions preserve affine cross-sections. The extended group coincides with the restriction $Aut(B^n_{\mathbb{C}}|_{B^n})$ to the real space of biholo. Möbius automorphisms of the complex unit ball.

- $\varphi_a : \{$ hyperplanes containing $a\} \leftrightarrows \{$ hyperplanes containing $0\}, \ |a| < 1,$
- $\varphi_{a^*} : \{$ hyperplanes containing $a\} \leftrightarrows \{$ hyperplanes parallel to $a\}, \ |a| > 1.$

- Computing pull-back integration measures (Jacobians) on the cross-sections of S^n by hyperplanes .

Kernel description

Theorem

Let $a \in \mathbb{R}^n$. Then $F_a f = 0,\ f \in C(S^{n-1})$ if and only if f is a-odd:

$$f(x) = -\rho_a(x) f(\tau_a x),\ x \in S^{n-1},$$

where

$$\rho_a(x) = \left(\frac{1-|a|^2}{|a-x|^2}\right)^{n-2},$$

and $\tau_a x$ is the a-reflection: $\tau_a x \in S^{n-1} \cap L(x,a)$.

Proof: intertwining with the standard Funk transform and referring to the known description of $kerF_0$.

Injectivity of paired Funk transform

For any $a, b \in \mathbb{R}^n \setminus S^{n-1}$, define $\Theta(a,b) = \frac{\langle a,b\rangle - 1}{\sqrt{(1-|a|^2)(1-|b|^2)}}$.

Theorem

Let $a, b \in \mathbb{R}^n$, $|a|, |b| \neq 1$. Then (F_a, F_b) fails to be injective, i.e., $Ker\ F_a \cap Ker\ F_b \neq \{0\}$, if and only if

1. $\Theta(a,b)$ *is real,*
2. $0 \leq \Theta(a,b) < 1$,
3. $\arccos \Theta(a,b) = \frac{p}{q}\pi$, *$p, q$ are integer.*

Geometric reformulation

Theorem (Equiv. geometric form)

The paired operator (F_a, F_b) is injective if and only if

1. $\langle a, b \rangle \neq 1$ *and*
2. *The straight line through a and b meets S^{n-1} and if not then* $\arccos \Theta(a, b) \neq \frac{p}{q}\pi$.

Reconstruction from the pairs of SFT

Reconstruction of the a-even part: if $F_a f = g_a$ then

$$f_a^+ = \frac{1}{2}(f + W_a f),$$

where $W_a f(x) = \rho_a(x) f(\tau_a x)$ and $\tau_a \in S^{n-1}$ is symmetric point to x w.r.t. a, $\rho_a(x)$ is the above defined weight function.

Let

$$(F_a f, F_b f) = (g_a, g_b).$$

Find from the inversion formulas:

$$f_a^+ = F_a^{-1} g_a := h_a, \quad f_b^+ = F_b^{-1} g_a := h_b.$$

Then write

$$f = 2h_a - W_a f, \; f = 2h_b - W_b f,$$

or

$$f = h + Wf,$$

where $h = 2h_a - 2W_a h_b$ and $W = W_a W_b$.

Reconstructing series

Iterate: $f = h + Wf = f + W(h + Wf) = f + Wh + W^2 f = \dots = \sum_{k=0}^{N-1} W^k h + W^N f$. Convergence: when $W^N f \to 0,\ N \to \infty$.

Theorem

Let $L(a,b) \cap S^{n-1} \neq \emptyset$. Then

$$f = \sum_{k=0}^{\infty} (W_a W_b)^k h,$$

where $W_a f(x) = \frac{1-|a|^2}{|a-x|^2} f(\tau_a x),\ W_b f(x) = \frac{1-|b|^2}{|b-x|^2} f(\tau_b x)$ *and* $h = 2(F_a^{-1} g_a - W_a F_b^{-1} g_b)$. *The series converges*

1. *uniformly on compact subsets of* $S^{n-1} \setminus \{a\}$ *or of* $S^{n-1} \setminus \{b\}$,
2. *in* $L^p(S^{n-1})$ *for* $1 \leq p < \frac{n}{n-1}$.

Injectivity of (F_a, F_b): key steps of the proof

Step 1. T- automorphic functions. If $f \in KerF_a \cap KerF_b$ then:

$$f(x) = -\rho_a(x) f(\tau_a x), \ (1)$$

$$f(y) = -\rho_b(y) f(\tau_b y). \ (2)$$

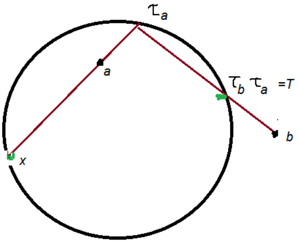
Substitute (1) to (2):

$$f(x) = \rho(x) f(Tx),$$

where $Tx = \tau_b \tau_a x$ the double reflection, $T : S^{n-1} \to S^{n-1}$ and $\rho(x) = \rho_b(\tau_a x)\rho_a(x), \ \rho_a(x) = \left(\frac{1-|a|^2}{|a-x|^2}\right)^{n-2}$.

Thus, $Ker\ F_a \cap Ker\ F_b$ consists of T-automorphic functions

The billiard $T : S^{n-1} \to S^{n-1}$



Step 2. The dynamics of the mapping T

- Reduction to the dynamics on the circle S^1 : The iterations $T^k x,\ k = 0, 1, \dots,\ Tx = \tau_b \tau_a x$, belong to $span(x,\ a,\ b)$.
- $T : S^1 \to S^1$ extends to a Moebius transformation $M_T \in SL(2, \mathbb{C})$.
- T-automorphic fnctns on S^{n-1} reduce to those on S^1 :

$$f(z) = |M'(z)|^{n-2} f(M_T(z)),\ z \in S^1.$$

Types of the orbits

Computation shows:

$$\det M_T = (|a|^2 - 1)(|b|^2 - 1),\ tr\ M_T = 2\frac{\langle a, b\rangle - 1}{\sqrt{(|a|^2 - 1)(|b|^2 - 1)}}.$$

The types of the dynamics (are determined by $tr\ M_T$):

- (1) Hyperbolic, parabolic, loxodromic: the orbits converge to an attractive fixed point on S^1.
- (2) Circular type: $tr\ M_T = 0$: the mapping of order 2 (periodic orbits).
- (3) Elliptic type: either M_T is of finite order (periodic orbits), if the rotation number $\theta(a, b) = \frac{p}{q}\pi$, or the orbits are dense, if $\theta(a, b) \neq \frac{p}{q}\pi$.)

Final step. Translation in terms of a, b.

$$tr\ M_T = \frac{2\langle a, b\rangle - 1}{\sqrt{(|a|^2 - 1)(|b|^2 - 1)}}.$$

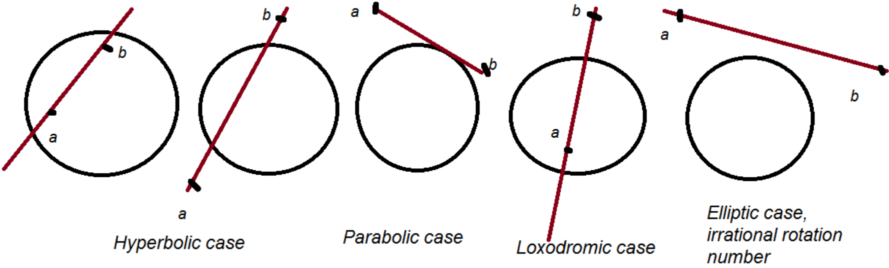
Lemma

There are nonzero M_T- automorphic fnctns if and only if the orbits are periodic, i.e., $M_T^N = id$, which happens in the

1. *circular case: $tr\ M_T = 0$, i.e. a, b are dual: $\langle a, b\rangle = 1$ and*
2. *elliptic case: $tr\ M_T \in (-1, 1)$, rotation number* $\arccos(\frac{1}{2} tr\ M_T) = \arccos \Theta(a, b) = \frac{p}{q}\pi$.

Theorem follows since, as we have shown before, $Ker\ (F_a, F_b)$ consists exactly of T-automorphic functions.

Injectivity cases



a and b are *not dual*: $< a , b > \neq 1$

Thank you for your attention!