# Support Theorems For Some Integral Transforms On Real-Analytic Riemannian Manifolds

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# Goal of the talk

**Goal** Establish **support theorems** for **integral transforms** like **T**ransverse **R**ay **T**ransform (**TRT**) and integral moments transform (of **G**eodesic **R**ay **T**ransform **(GRT)**)

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But, first...

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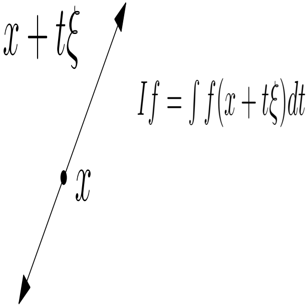

Q2 What is a support theorem?

Let $f$ be a function (or a tensor field) which is compactly supported in a Riemannian manifold $M$ (with smooth boundary). Let $If$ denote any particular integral transform, e.g. an X-ray transform. Assume that $If$ is zero over a connected open set $\mathcal{A}$ of submanifolds $\Xi$ of the manifold $M$. One then tries to prove that $f$ vanishes over $\bigcup_{\Xi \in \mathcal{A}} \Xi$.

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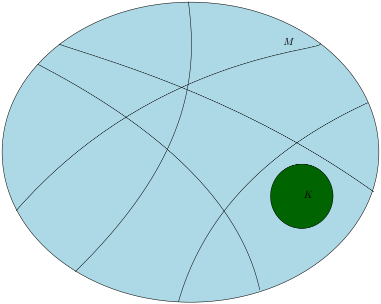

# Definitions and notations

- $(M, g)$ represents a compact, **simple**, real-analytic Riemannian manifold of dimension $n$ with smooth boundary
- For $x \in \partial M$, $\gamma_{x,\xi}(t)$ is the geodesic starting from $x$ in the direction $\xi$ and $l(\gamma_{x,\xi})$ is the value of $t$ at which this geodesic hits the boundary again
- $K \subset M$ is said to be geodesically convex if for any two points $x \in K$ and $y \in K$, the geodesic connecting them lies entirely in the set $K$
- Will use Einstein summation convention (summing over repeated indices). Furthermore, we will use coordinate representations, an $m$- tensor field $f(x)$ will be written as $f_{i_1 \dots i_m} dx^{i_1} \otimes \cdots \otimes dx^{i_m}$ where $f_{i_1 \dots i_m}$ is the corresponding coordinate function for basis vector $dx^{i_1} \otimes \cdots \otimes dx^{i_m}$

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- Consider $\widetilde{M}$ to be a real analytic extension of $M$. Let $\mathcal{A}$ be an open set of geodesics with endpoints in $\widetilde{M} \setminus M$ such that any geodesic in $\mathcal{A}$ is homotopic, within the set $\mathcal{A}$, to a geodesic lying outside $M$. Set of points lying on the geodesics in $\mathcal{A}$ is denoted by $M_{\mathcal{A}}$ i.e.

$$M_{\mathcal{A}} = \bigcup_{\gamma \in \mathcal{A}} \gamma$$

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# Support theorem for TRT

# Transverse ray transform

- **Definition**: $f \in C_c^\infty(M)$ is an $m$- tensor field, $\gamma(t)$ is a geodesic, $\eta(t)$ is a vector field formed by parallel translation along $\gamma(t)$ and orthogonal to it, then TRT is defined by:

$$Jf(\gamma, \eta) = \int_0^{l(\gamma)} f_{i_1 \dots i_m}(\gamma(t))\eta^{i_1} \dots \eta^{i_m} dt \quad (\eta(t) \in \gamma^\perp(t))$$

- Motivation: Polarization tomography [Sharafutdinov]
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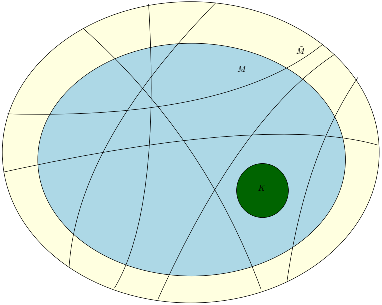


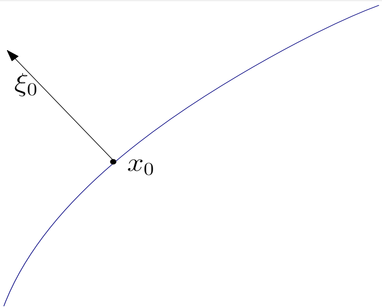

**Theorem 1 [A.,2019]**

Let $(M, g)$ be a simple real analytic Riemannian manifold of dimension $n \geq 3$ and $\widetilde{M}$ be a real analytic extension of $M$. Let $\mathcal{A}$ be any connected open set of geodesics as before. Let $f \in \mathcal{E}'(\widetilde{M})$ be a symmetric $m$- tensor field supported in $M$. If $Jf(\gamma, \eta) = 0$ for every $\gamma \in \mathcal{A}$ and for every $\eta \in \gamma^{\perp}$, then $f = 0$ on $M_{\mathcal{A}}$.

# Key steps in the proof

**A microlocal proposition [A.,2018]**

Let $(x_0, \xi_0) \in T^*M \setminus 0$ and let $\gamma_0$ be a fixed simple geodesic through $x_0$ normal to $\xi_0$.



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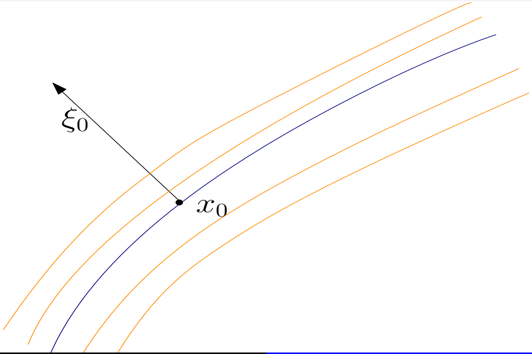

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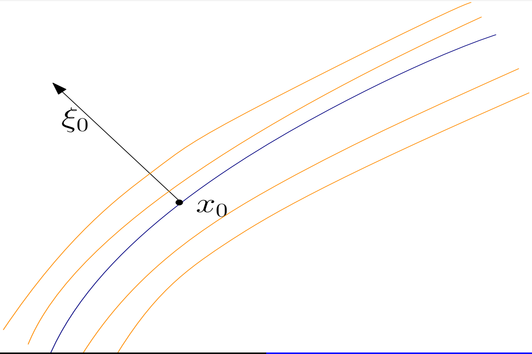

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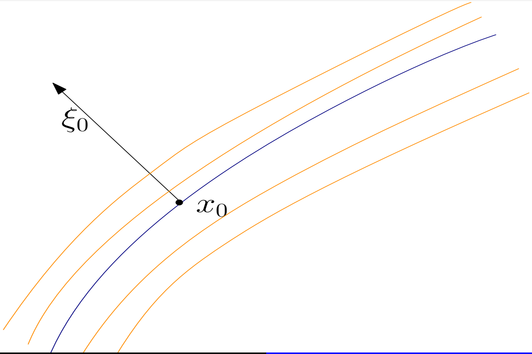

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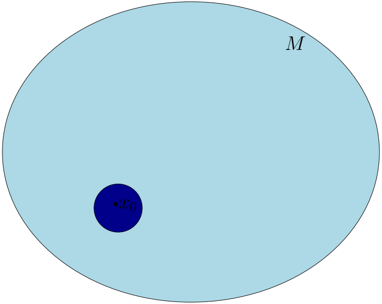
Let $f \in \mathcal{D}'(M)$. Let $x_0 \in M$ and let $U$ be a neighborhood of $x_0$.



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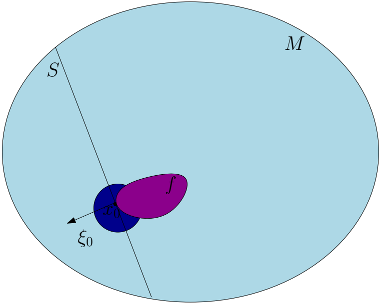

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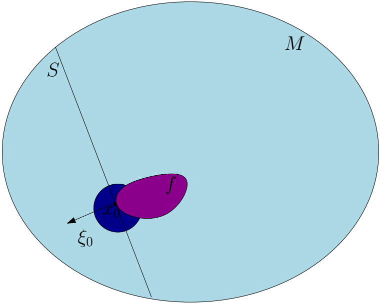
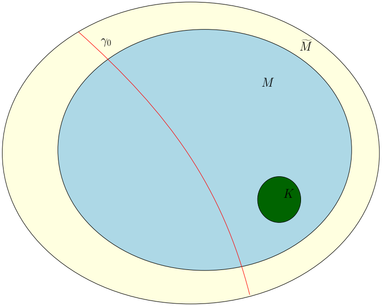


# A "support diminishing" argument

- $\gamma_0$ can be continuously deformed to $\gamma_1$ while remaining within $\mathcal{A}$, intermediate geodesics in this deformation are $\gamma_t$



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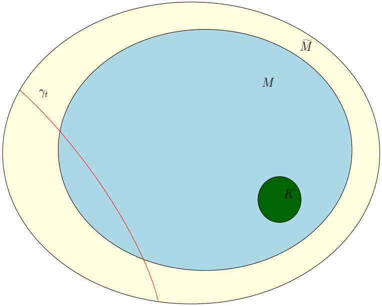

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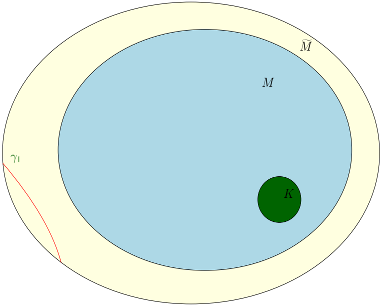

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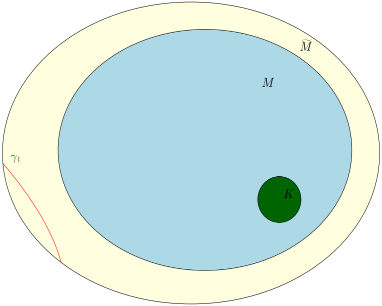

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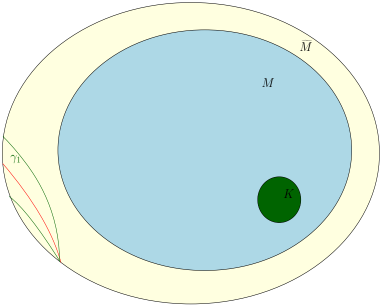

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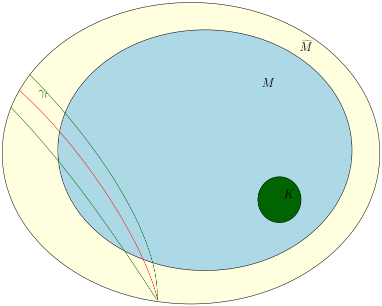

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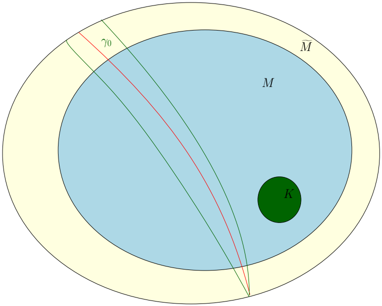

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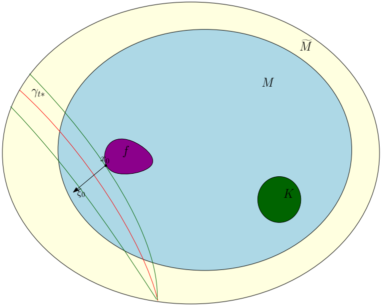

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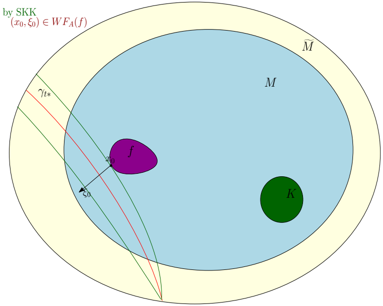

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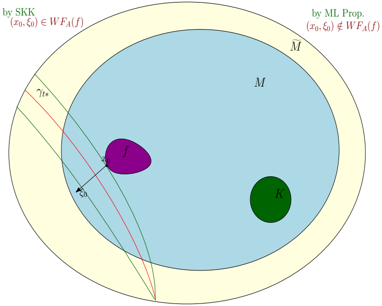

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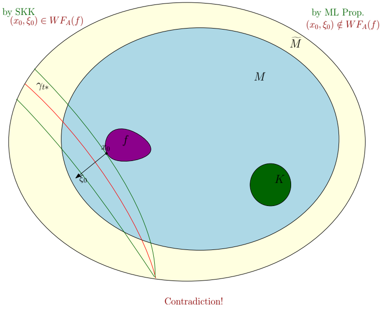

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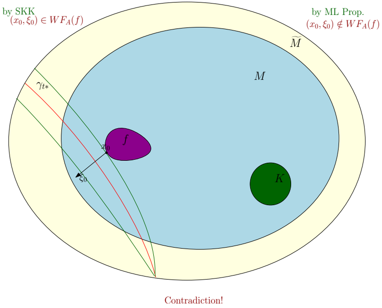
- Arguments of this kind go back to [Boman, Quinto (1987)]



# Remarks on proof of the microlocal proposition

- Writing in local co-ordinates,
  $(x_0, \xi_0) \notin WF_A(f) \equiv (x_0, \xi_0) \notin WF_A(f_{i_1 \dots, i_m})$
- Use generalized **FBI Transform** characterisation for analytic wavefront sets: For $u \in \mathcal{D}'(\mathbb{R}^n)$, and $\phi(x, y)$ is holomorphic in a complex neighbourhood of $(x_0, y_0) \in \mathbb{C}^n \times \mathbb{R}^n$ such that:

$$\Im(\partial_y^2 \phi)(x_0, y_0) > 0; \quad (\det(\partial_x \partial_y)\phi)(x_0, y_0) \neq 0$$

  then

$$T^*(\mathbb{R}^n) \setminus 0 \ni (x_0, y_0) \notin WF_A(u)$$

  iff

$$\int e^{i\phi(x,y)/h} u(y) dy = \mathcal{O}(e^{-\delta/h})$$

  uniformly for $(x, y)$ in a neighbourhood of $(x_0, y_0)$ and as $h > 0$ goes to 0.

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# Remarks (contd.)

- Starting with $Jf(\gamma, \eta_1, \ldots, \eta_m) = 0$ for $\gamma$ near $\gamma_0$, some algebraic manipulations later we get:

$$\iint e^{i\Phi(y,x,\xi,v)/h} a_N(x,\xi) f_{i_1 \cdots m}(x) b^{i_1}(x,\xi) \ldots b^{i_m}(x,\xi) dx d\xi = 0$$

$\Phi = (x-y)\xi + i(\xi - v)^2/2$, $a_N, b$ are analytic in $(x, \xi)$

- Split the $x$-integral above into two parts- $I_1$ contains no critical points of $\Phi$, and $I_2$ containing exactly one critical point of $\Phi$ (w.r.t $\xi$). $I_1$ is then analysed using an integration by parts argument and $I_2$ is estimated using stationary phase method.
- More algebraic manipulation later we get [A.,2018]: $\int e^{i\psi(x,\xi)/h} f_{i_1 \ldots i_m}(x) B^{i_1 \cdots i_m}(x, \xi; h) dx = \mathcal{O}(e^{-\delta/h})$ where $\psi(x,\xi)$ is a function with same properties as in the definition of FBI transform and $B^{i_1 \ldots i_m}$ is a classical analytic symbol with $\sigma_p(B)^{ij}(0, \xi_0) = \eta^{i_1} \ldots \eta^{i_m}$. This would prove the microlocal proposition.

# Remarks (contd.)

- Starting with $Jf(\gamma, \eta_1, \dots, \eta_m) = 0$ for $\gamma$ near $\gamma_0$, some algebraic manipulations later we get:

$$\iint e^{i\Phi(y,x,\xi,v)/h} a_N(x,\xi) f_{i_1 \cdots m}(x) b^{i_1}(x,\xi) \dots b^{i_m}(x,\xi) dx d\xi = 0$$

$\Phi = (x-y)\xi + i(\xi - v)^2/2$, $a_N, b$ are analytic in $(x,\xi)$

- Split the $x$-integral above into two parts- $I_1$ contains no critical points of $\Phi$, and $I_2$ containing exactly one critical point of $\Phi$ (w.r.t $\xi$). $I_1$ is then analysed using an integration by parts argument and $I_2$ is estimated using stationary phase method.
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## Support theorem for integral moments of GRT

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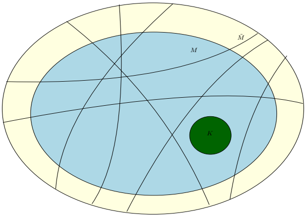
- $q$-th integral moment of a symmetric $m$-tensor field $f$, $I^q f$ is a function defined by
$I^q f(x,\xi) = \int_0^{l(\gamma_{x,\xi})} t^q f_{i_1\ldots i_m}(\gamma_{x,\xi}(t))\dot{\gamma}^{i_1}_{x,\xi}(t)\cdots\dot{\gamma}^{i_m}_{x,\xi}(t)dt$

**Support theorem for integral moments [A.,Mishra (2017)]**

Let $f$ be a symmetric $m$-tensor field on a manifold as above with components in $\mathcal{E}'(\widetilde{M})$ where $\widetilde{M}$ is an extension of $M$ and $K$ be a closed geodesically convex subset of $M$. If for each geodesic $\gamma$ not intersecting $K$, we have that $I^q f(\gamma) = 0$ for $q = 0, 1, \ldots, m$ then $supp(f) \subset K$.



# Why do we need integral moments of GRT?

**Decomposition Theorem [Sharafutdinov]**

Let $M$ be a compact Riemannian manifold with boundary; let $k \geq 1$ and $m \geq 0$ be integers. For every field $f \in H^k(S^m(M))$, there exist uniquely determined $f^s \in H^k(S^m(M))$ and $v \in H^{k+1}(S^{m-1}(M))$ such that

$$f = f^s + dv, \qquad \delta f^s = 0, \qquad v|_{\partial M} = 0.$$

Here $\delta$ is the divergence operator and $dv$ represents the symmetrised covariant derivative of $v$.

- Writing $u_{i_1 \dots i_m}(\gamma(t))\dot{\gamma}^{i_1} \dots \dot{\gamma}^{i_m} := \langle u(\gamma(t)), \dot{\gamma}^{\otimes m} \rangle$ , we can verify the identity: $\frac{d}{dt}\langle v(\gamma(t)), \dot{\gamma}(t)^{\otimes m-1} \rangle = \langle (dv(\gamma(t))), \dot{\gamma}(t)^{\otimes m} \rangle$
- A tensor field of the form $f = dv$ such that $v|_{\partial M} = 0$ lies in the kernel of $I^0$. This indicates additional information (e.g., integral moments) is required to prove a support theorem.

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# Key steps in the proof

**Theorem 1 [A.,Mishra (2017)]**

Let $f$ be a symmetric $m$-tensor field with components in $\mathcal{E}'(\widetilde{M})$ and $K$ be a closed geodesically convex subset of $M$. If for each geodesic $\gamma$ not intersecting $K$, we have that $I^0 f(\gamma) = 0$, then we can find an $(m-1)$-tensor field $v$ with components in $\mathcal{D}'(\text{int}(\widetilde{M}) \setminus K)$ such that $f = dv$ in $\text{int}(\widetilde{M}) \setminus K$ and $v = 0$ in $\text{int}(\widetilde{M}) \setminus M$. (Krishnan and Stefanov proved this for 2- tensor fields)

- *Lemma* [A.,Mishra] : For any $1 \leq k \leq m$, if $f = dv$ with $v|_{\partial M} = 0$. Then $I^k f = -kI^{k-1}v$.
- By using the above two iteratively, we get: $I^m f(\gamma) = m!(-1)^m I^0 v_m(\gamma) = 0$, where $v_m$ is distribution.

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**Theorem [Krishnan]**

Assume $(M, g)$ is a manifold as above and $K$ is a geodesically convex subset of $M$. If for a distribution $u \in \mathcal{E}'(M)$, $I^0 u(\gamma) = 0$ for each geodesic $\gamma$ not intersecting $K$, then $u = 0$ outside $K$.

Using these results, we get the proof for the support theorem for integral moments.

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**Thank You!**