A uniqueness result for a light ray transform of symmetric two tensor field

Manmohan Vashisth

mvashisth@csrc.ac.cn

Beijing Computational Science Reserach Center
Beijing, China

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Notation and convention

- $\Omega =$ bounded open domain in $\mathbb{R}^{1+n}$ with $n \geq 3$ and for $z := (t, x) \in \Omega$, $\nabla_z := (\partial_t, \nabla_x) := (\partial_0, \partial_1, \partial_2, \cdots, \partial_n)$
- For repeating indices, we assume the Einstein summation notation, and also the convention that $\theta^0 = 1$.
- $F = ((F_{ij}))_{0 \leq i,j \leq n}$ is symmetric 2-tensor and

$$\delta F = (\partial_j F_{0j}, \partial_j F_{1j}, \partial_j F_{2j}, \cdots, \partial_j F_{nj})^T$$

- For a vector field $v = (v_0, v_1, v_2, \cdots, v_n)$; the symmetried derivative d of v is given by the following matrix:

$$dv = \left(\left(\frac{\partial_i v_j + \partial_j v_i}{2}\right)\right)_{0 \leq i,j \leq n}$$

Light ray transform

Let $f = ((f_{i_1 i_2 \cdots i_m}))_{\{0 \le i_j \le n;\ 1 \le j \le m\}}$ be an m-tensor field and then its light ray transform Lf at $(t, x) \in \mathbb{R}^{1+n}$ in the direction of $(1, \theta)$ is defined by

$$Lf(t, x; \theta) := \int\limits_{\mathbb{R}} \theta^{i_1} \theta^{i_2} \cdots \theta^{i_m} f_{i_1 i_2 \cdots i_m}(t + s, x + s\theta) \mathrm{d}s \tag{1}$$

where $\theta \in \mathbb{S}^{n-1}$ and $(t, x) \in \mathbb{R}^{1+n}$.

- For $m = 0$ (function case) Light ray transform appear in determining the time-dependent potential appearing in hypebolic PDE from boundary or scattering data; see for example: Stefanov (1989), Waters (2014), Ben Aïcha (2015), Kian (2016), Oksanen-Kian (2016) and several others..........

Light ray transform

- For $m = 1$ (vector field case) these transform appears in determination of first order time-dependent perturbation in hyperbolic equations from boundary data; see for example Salazar (2013), Montalto (2014), Stefanov-Yang (2018), Krishnan-Vashisth (2018), Feizmohammadi-Ilmavirta-Kian-Oksanen (2019) and many more.........
- For $m = 2$ (2-tensor case) these trasform appears in determining the time-dependent coefficients of quadratic non-linearity in Non-linear hyperbolic PDE; see for example Nakamura-Vashisth (2017)

Light ray transforms for symmetric m-tensor for $m = 0, 1, 2$ in Euclidean and Lorentzian geometry have been studied:

Stefanov, Lassas, Oksanen, Uhlmann, Wang, RabieniaHaratbar, Waters-Salazar and several others......

Light ray 2-tensor

Let $F(t,x) = ((F_{ij}(t,x)))_{0\leq i,j\leq n}$ be a symmetric 2-tensor field defined on Ω and we extend it by zero outside Ω, then light ray transform LF of F is defined by

Light Ray 2-tensor

$$LF(t,x;\theta) := \int_{\mathbb{R}} \theta^i\theta^j F_{ij}(t+s, x+s\theta),\ (t,x) \in \mathbb{R}^{1+n} \text{and } \theta \in \mathbb{S}^{n-1}.$$

Kernel of L

For $\lambda \in C^\infty(\Omega)$ function, g is the Minkowski metric with $(-1,1,1,\cdots,1)$ along the diagonal and a smooth vector-field v satisfying $v|_{\partial\Omega} = 0$, we have

$$L(\lambda g + \mathrm{d}v)(t,x,\theta) = 0, \text{ for all } (t,x) \in \mathbb{R}^{1+n} \text{ and } \theta \in \mathbb{S}^{n-1}.$$

Light ray 2-tensor

Problem of interest

If $LF(t,x,\theta)=0$, for all $(t,x)\in\mathbb{R}^{1+n}$ and $\theta\in\mathbb{S}^{n-1}$ near some fixed $\pm\theta_0\in\mathbb{S}^{n-1}$, then can we characterize such symmetric 2-tensor fields?

In this talk, we show that only symmetric two tensor satisfying $LF(t,x,\theta)=0$ are of $\lambda g+dv$ form. More precisely, we prove the following:

Main Theorem

Theorem (Krishnan-Senapati-V.)

Let $F \in C^\infty(\Omega)$ be a symmetric 2-tensor field. If for a fixed $\theta_0 \in \mathbb{S}^{n-1}$,

$$LF(t,x,\theta) = 0, \quad \text{for all } (t,x) \in \mathbb{R}^{1+n} \quad \text{and} \quad \theta \text{ near } \pm\theta_0,$$

then $F = \lambda g + \mathrm{d}v$, where λ is a C^∞ function, g is the Minkowski metric with $(-1,1,1,\cdots,1)$ along the diagonal, v is a C^∞ vector field with $v|_{\partial\Omega} = 0$, and d is the symmetrized derivative.

Decomposition theorem

Theorem (Krishnan-Senapati-V.)

Let $F \in C^\infty(\Omega)$ be a symmetric 2-tensor field. Then there exists a symmetric 2-tensor field $\widetilde{F}$ satisfying $\delta(\widetilde{F}) = trace(\widetilde{F}) = 0$, a C^∞ function λ and a vector field v satisfing $v|_{\partial\Omega} = 0$ such that F can be decomposed as

$$F = \widetilde{F} + \lambda g + \mathrm{d}v. \tag{2}$$

Here g is the Minkowski metric with $(-1, 1, 1, \cdots, 1)$ along the diagonal and d is the symmetrized derivative of v defined by

$$(\mathrm{d}v)_{ij} = \frac{1}{2}\left(\partial_i v_j + \partial_j v_i\right).$$

- Analogous to the above decomposition theorem in Riemannian geometry is proved by Sharafutdinov (2007).

Uniqueness for trace and divergence free tensor

Assuming that the decomposition theorem is true, then we have

$$LF(t,x,\theta) = L\widetilde{F}(t,x,\theta), \text{for all } (t,x) \in \mathbb{R}^{1+n} \text{ and } \theta \in \mathbb{S}^{n-1}$$

where $\widetilde{F}$ is as in the decomposition theorem. Therefore, it is enough to prove the following:

Theorem (Krishnan-Senapati-V.)

Let $F \in C^{\infty}(\Omega)$ be a symmetric 2-tensor field with $\delta F = 0$ and trace$(F) = 0$. *If for a fixed $\theta_0 \in \mathbb{S}^{n-1}$,*

$$LF(t,x,\theta) = 0, \quad \textit{for all } (t,x) \in \mathbb{R}^{1+n} \quad \textit{and} \quad \theta \textit{ near } \pm\theta_0,$$

then $F = 0$.

Sketch of proof

We follow the arguments similar to the one used in Stefanov (2017), RabieniaHaratbar (2018), Krishnan and Vashisth (2018); let

- $z = (t, x) \in \mathbb{R}^{1+n}$ and $\nabla_z = (\partial_t, \nabla_x) := (\partial_0, \partial_1, \partial_2, \cdots, \partial_n)$
- $\omega \in \mathbb{R}^{1+n}$ is arbitrary

we have

$$(\omega \cdot \nabla_z)(LF)(t, x, \theta) = \int_{\mathbb{R}} \theta^i \theta^j \omega^k \partial_k F_{ij}(t + s, x + s\theta)\, \mathrm{d}s \qquad (3)$$

holds for all $\omega \in \mathbb{R}^n$, $(t, x) \in \mathbb{R}^{1+n}$ and $\theta \in \mathbb{S}^{n-1}$. Also by fundamental theorem of calculus, we have

Sketch of proof

$$
\begin{aligned}
0 &= \int_{\mathbb{R}} \frac{d}{ds}(\theta^i \omega^k F_{ik})(t+s, x+s\theta)\,\mathrm{d}s \\
&= \int_{\mathbb{R}} \theta^i \theta^j \omega^k \partial_j F_{ik}(t+s, x+s\theta)\,\mathrm{d}s
\end{aligned} \tag{4}
$$

Subtracting (4) from (3) and using the hypothesis

- $LF(t, x, \theta) = 0$ for θ near $\pm\theta_0$,

$$
\int_{\mathbb{R}} \theta^i \theta^j \omega^k \left(\partial_k F_{ij} - \partial_j F_{ik}\right)(t+s, x+s\theta)\,\mathrm{d}s = 0,
$$

holds for all $(t, x) \in \mathbb{R}^{1+n}, \theta \approx \pm\theta_0$.

Sketch of proof

Denote $h_{ijk} := \partial_k F_{ij} - \partial_j F_{ik}$, we have

$$Ih(t,x;\theta,\omega) := \int_{\mathbb{R}} \theta^i \theta^j \omega^k h_{ijk}(t+s, x+s\theta)\,\mathrm{d}s = 0; \tag{5}$$

holds for all $(t,x) \in \mathbb{R}^{1+n}$, $\omega \in \mathbb{R}^{1+n}$ and θ near $\pm\theta_0$. Next consider the Fourier transform:

$$\widehat{h}_{ijk}(\zeta) = \int_{\mathbb{R}^{1+n}} h_{ijk}(t,x) e^{-\mathrm{i}(t,x)\cdot\zeta}\,\mathrm{d}t\mathrm{d}x \tag{6}$$

Using the decomposition,

$\mathbb{R}^{1+n} = \mathbb{R}(1,\theta) \oplus \ell$ with $\ell \in (1,\theta)^{\perp}$ combined with Fubini's theorem,

Sketch of proof

we get

$$\widehat{h}_{ijk}(\zeta) = \sqrt{2} \int\limits_{(1,\theta)^\perp} \int\limits_{\mathbb{R}} \theta^i \theta^j \omega^k h_{ijk}(\ell + s(1,\theta)) e^{-i(\ell + s(1,\theta))\cdot\zeta} \, \mathrm{d}s \, \mathrm{d}\ell$$

If $\zeta \in (1,\theta)^\perp$, then

$$\theta^i \theta^j \omega^k \widehat{h}_{ijk}(\zeta) = \sqrt{2} \int\limits_{(1,\theta)^\perp} \int\limits_{\mathbb{R}} \theta^i \theta^j \omega^k h_{ijk}(s(1,\theta) + \ell) e^{-i\ell\cdot\zeta} \, \mathrm{d}s \, \mathrm{d}\ell.$$

Using (5), we get that

$$\theta^i \theta^j \omega^k \widehat{h}_{ijk}(\zeta) = 0; \text{ for all } \omega \in \mathbb{R}^{1+n}, \; \zeta \in (1,\theta)^\perp \text{ with } \theta \approx \pm\theta_0.$$

Sketch of proof

Finally choosing $\omega = e_1 = (1, 0, 0, \cdots, 0) \in \mathbb{R}^{1+n}$ and definition of h_{ijk}, we get

$$\theta^i \theta^j \widehat{F}_{ij}(\zeta) = 0 \text{ for all } \zeta \in (1, \theta)^{\perp} \text{ and } \theta \text{ near } \pm \theta_0. \tag{7}$$

From here we want to show that $F_{ij} \equiv 0$ for all $0 \leq i, j \leq n$.

Idea for proof

First show that $\widehat{F}_{ij}(\zeta_0) = 0$ for $\zeta_0 := e_2 = (0, 0, 1, \cdots, 0) \in \mathbb{R}^{1+n}$ fixed space-like vector and for all $0 \leq i, j \leq n$. Then we show that $\widehat{F}_{ij}(\zeta) = 0$, for all space-like vector ζ near ζ_0 hence finally using the Paley-Wiener thereom, we conclude that $F_{ij}(t, x) = 0$ in Ω for all $0 \leq i, j \leq n$.

We give the sketch of proof for $n = 3$ and similar idea can be used for $n \geq 4$.

Proof for $n = 3$ special case

- Fix $\zeta_0 = (0, 0, 1, 0) \in \mathbb{R}^{1+3}$ and $\pm\theta_0 = (\pm 1, 0, 0)$.
- $(1, \pm\theta_0) \cdot \zeta_0 = 0$

Now consider

$$\pm\,\theta_0(a) = (\pm\cos a, 0, \sin a). \tag{8}$$

- If a is near 0, then $\pm\theta_0(a)$ is near $\pm\theta_0$.
- Also $(1, \pm\theta_0(a)) \cdot \zeta_0 = 0$.

Proof for $n = 3$ special case

Using this choice of ζ_0 and $\pm\theta_0(a)$ in

$$\theta^i \theta^j \widehat{F}_{ij}(\zeta) = 0 \text{ for all } \zeta \in (1, \theta)^{\perp} \text{ and } \theta \text{ near } \pm \theta_0$$

we get

$$\begin{aligned}\Big(&\widehat{F}_{00} \pm 2\cos a \widehat{F}_{01} + 2\sin a \widehat{F}_{03} + \cos^2 a \widehat{F}_{11} \\ &\pm 2\sin a \cos a \widehat{F}_{13} + \sin^2 a \widehat{F}_{33}\Big)(\zeta_0) = 0, \text{ for } a \text{ near } 0.\end{aligned} \tag{9}$$

Proof for $n = 3$ special case

Now Differentiating above equation twicely w.r.t. a and taking $a \to 0$, we get the following set of equations

$$\left(\widehat{F}_{00} \pm 2\widehat{F}_{01} + \widehat{F}_{11}\right)(\zeta_0) = 0$$
$$\left(\widehat{F}_{03} \pm \widehat{F}_{13}\right)(\zeta_0) = 0$$
$$\left(\mp\widehat{F}_{01} - \widehat{F}_{11} + \widehat{F}_{33}\right)(\zeta_0) = 0.$$

Consider the above equations with the positive and negative signs separately, adding and substracting, we get the following five equations:

$$\begin{aligned}&\left(\widehat{F}_{00} + \widehat{F}_{11}\right)(\zeta_0) = 0;\ \ \widehat{F}_{01}(\zeta_0) = 0;\ \ \widehat{F}_{03}(\zeta_0) = 0;\\ &\widehat{F}_{13}(\zeta_0) = 0; \left(-\widehat{F}_{11} + \widehat{F}_{33}\right)(\zeta_0) = 0\end{aligned} \tag{10}$$

Proof for $n = 3$ special case

Since $\delta(F) = \text{trace}(F) = 0$, we have

$$\begin{aligned} &\widehat{F}_{02}(\zeta_0) = \widehat{F}_{12}(\zeta_0) = \widehat{F}_{22}(\zeta_0) = \widehat{F}_{32}(\zeta_0) = 0, \\ &\left(\widehat{F}_{00} + \widehat{F}_{11} + \widehat{F}_{22} + \widehat{F}_{33}\right)(\zeta_0) = 0. \end{aligned} \tag{11}$$

From these 10 equations in (10) and (11), we get $\widehat{F}_{ij}(\zeta_0) = 0$ for all $0 \leq i, j \leq 3$.

Proof for $n = 3$ general case

- Next, our goal is to show that $F_{ij}(\zeta) = 0$, for $\zeta \neq 0$ space-like vector in a small enough conical neighborhood of ζ_0.

We start with a unit vector in $\mathbb{R}^3$, $\zeta' := (\zeta^1, \zeta^2, \zeta^3) \in \mathbb{S}^2$, and let us choose $\zeta^0 = -\sin\varphi$. Then $(-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$ is a space-like vector if $-\pi/2 < \varphi < \pi/2$.

- Let us recall that in showing $F_{ij}(\zeta_0) = 0$, we considered a perturbation $\pm\theta_0(a)$ (see (8)) of the vector $\pm\theta_0 = (\pm 1, 0, 0)$. Note that we required that $\pm\theta_0(a)$ was close enough to $\pm\theta_0$ and $(1, \pm\theta_0(a)) \cdot \zeta_0 = 0$. Our next calculations are motivated by these requirements for the vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$.

Proof for $n = 3$ general case

Since we are interested in a non-zero space-like vector in a small enough conical neighborhood of ζ_0, let us choose

$$\zeta^1 = \sin\alpha\cos\beta,\ \zeta^2 = \cos\alpha \text{ and } \zeta^3 = \sin\alpha\sin\beta.$$

Then clearly ζ is close to $(0,1,0)$ whenever α and β are close enough to 0, and choosing φ close to 0, we get that the space-like vector $\zeta = (-\sin\varphi, \zeta^1, \zeta^2, \zeta^3)$ is close enough to $(0,0,1,0)$.

- Next choose $\pm\theta_0(\varphi) := (\pm\cos\varphi, \sin\varphi, 0)$ close to $\pm\theta_0$ when φ is close to 0 and the perturbation of $\theta_0(\varphi)$ (for a close to 0) by

$$\pm\theta_0(\varphi, a) = (\pm\cos a\cos\varphi, \sin\varphi, \sin a\cos\varphi).$$

Proof for $n = 3$ general case

Let us consider the orthogonal matrix A:

$$A = \begin{bmatrix} \cos\alpha\cos\beta & -\sin\alpha & \cos\alpha\sin\beta \\ \sin\alpha\cos\beta & \cos\alpha & \sin\alpha\sin\beta \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

Define $\pm\widetilde{\theta}_0$, $\pm\widetilde{\theta}_0(\varphi)$ and $\pm\widetilde{\theta}_0(a,\varphi)$ by

$$\pm\widetilde{\theta}_0 := A^T \begin{bmatrix} \pm 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \pm\cos\alpha\cos\beta \\ \mp\sin\alpha \\ \pm\cos\alpha\sin\beta \end{bmatrix}$$

$$\pm\widetilde{\theta}_0(\varphi) = A^T\left(\pm\theta_0(\varphi)\right) = A^T \begin{bmatrix} \pm\cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \pm a_{11}\cos\varphi + a_{21}\sin\varphi \\ \pm a_{12}\cos\varphi + a_{22}\sin\varphi \\ \pm a_{13}\cos\varphi + a_{23}\sin\varphi \end{bmatrix}$$

Proof for $n = 3$ general case

$$\pm\,\widetilde{\theta}_0(\varphi, a) = A^T\left(\pm\theta_0(\varphi, a)\right)$$
$$= \begin{bmatrix} \pm a_{11}\cos a\cos\varphi + a_{21}\sin\varphi + a_{31}\sin a\cos\varphi \\ \pm a_{12}\cos a\cos\varphi + a_{22}\sin\varphi + a_{32}\sin a\cos\varphi \\ \pm a_{13}\cos a\cos\varphi + a_{23}\sin\varphi + a_{33}\sin a\cos\varphi \end{bmatrix} = \begin{bmatrix} A_1^{\pm}(a) \\ A_2^{\pm}(a) \\ A_3^{\pm}(a) \end{bmatrix}.$$

- Note that if a, φ, α and β are close enough to 0, then $\pm\widetilde{\theta}_0(\varphi, a) \approx \pm\theta_0$. Therefore $LF(t, x, \pm\widetilde{\theta}_0(\varphi, a)) = 0$.
- Also note that for all φ, a, α and β close enough to 0, $(1, \pm\widetilde{\theta}_0(\varphi, a)) \cdot \zeta = 0$.

Proof for $n = 3$ general case

Therefore, using these choice of ζ and $\pm\widetilde{\theta}_0(\varphi, a)$ in

$$\theta^i\theta^j\widehat{F}_{ij}(\zeta) = 0 \text{ for all } \zeta \in (1,\theta)^\perp \text{ and } \theta \text{ near } \pm\theta_0$$

we get that

$$A_i^{\pm}(a)A_j^{\pm}(a)\widehat{F}_{ij}(\zeta) = 0; \text{ for } a \text{ near to } 0.$$

- Differentiating above equation twicely w.r.t. a and taking $a \to 0$
- Consider the two equations corresponding to the positive and negative signs and adding and substracting them, we get five set of equations which coincides with the five equations in (10) as $\alpha, \beta, \varphi \to 0$.

Proof for $n = 3$ general case

- Divergence free and trace free conditions give us five more equations, which are again identical with the five equations in (11) as $\alpha, \beta, \varphi \to 0$.
- Since we know that $F_{ij}(\zeta_0) = 0$ for $0 \leq i, j \leq 3$, we have that the determinant of the matrix formed by the 10 equations in (10) and (11) is non-zero.
- Therefore we have $\widehat{F}_{ij}(\zeta) = 0$ for $0 \leq i, j \leq 3$, where $\zeta = (-\sin\varphi, \sin\alpha\cos\beta, \cos\alpha, \sin\alpha\sin\beta)$, where α, β and φ are near 0. By the same argument $\widehat{F}_{ij}(\lambda\zeta) = 0$ for $0 \leq i, j \leq 3$, where ζ is as above and $\lambda > 0$.

So by using the Paley-Wiener theorem, we conclude that $F \equiv 0$ in Ω. This completes the proof for $n = 3$. Proof for $n \geq 4$ dimensions follows by using the similar arguments.

Proof for the decomposition theorem

Assume that the decomposition is true. Then

$$\operatorname{trace}(F) = \operatorname{trace}(\widetilde{F}) + \operatorname{trace}(\lambda g) + \operatorname{trace}(\mathrm{d}v).$$
$$\delta F = \delta(\widetilde{F}) + \delta\,(\lambda g) + \delta \mathrm{d}v.$$

Using $\operatorname{trace}(\widetilde{F}) = \delta(\widetilde{F}) = 0$, $\operatorname{trace}(\lambda g) = (n-1)\lambda$, $\delta(\lambda g) = (-\partial_0\lambda, \partial_1\lambda, \partial_2\lambda, \cdots, \partial_n\lambda)^T$ and $\operatorname{trace}(\mathrm{d}v) = \delta v$, we have

$$\operatorname{trace}(F) = (n-1)\lambda + (\partial_0 v_0 + \partial_1 v_1 + \partial_2 v_2 + \cdots + \partial_n v_n)\,. \quad (12)$$

Proof for the decomposition theorem

$$\begin{bmatrix} \partial_j F_{0j} \\ \partial_j F_{1j} \\ \partial_j F_{2j} \\ \vdots \\ \partial_j F_{nj} \end{bmatrix} = \begin{bmatrix} -\partial_0 \lambda \\ \partial_1 \lambda \\ \partial_2 \lambda \\ \vdots \\ \partial_n \lambda \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \Delta v_0 + \partial^2_{0j} v_j \\ \Delta v_1 + \partial^2_{1j} v_j \\ \Delta v_2 + \partial^2_{0j} v_j \\ \vdots \\ \Delta v_n + \partial^2_{nj} v_j \end{bmatrix} \tag{13}$$

Now using λ from (12) in (13), we get

Proof for the decomposition theorem

$$\begin{bmatrix} \Delta v_0 + \alpha\partial^2_{0j} v_j \\ \Delta v_1 + \beta\partial^2_{1j} v_j \\ \Delta v_2 + \beta\partial^2_{0j} v_j \\ \vdots \\ \Delta v_n + \beta\partial^2_{nj} v_j \end{bmatrix} = \begin{bmatrix} \partial_j f_{0j} \\ \partial_j f_{1j} \\ \partial_j f_{2j} \\ \vdots \\ \partial_j f_{nj} \end{bmatrix} - \frac{1}{n-1} \begin{bmatrix} -\partial_0 \mathrm{tr}(f) \\ \partial_1 \mathrm{tr}(f) \\ \partial_2 \mathrm{tr}(f) \\ \vdots \\ \partial_n \mathrm{tr}(f) \end{bmatrix} := \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{14}$$

where $\alpha := \left(1 + \frac{2}{n-1}\right)$ and $\beta := \left(1 - \frac{2}{n-1}\right)$. Existence of v can be proved by solving the above system of equations.

Proof for $n = 3$

$$
\begin{cases}
3\partial_0^2 v_0 + \partial_1^2 v_0 + \partial_2^2 v_0 + \partial_3^2 v_0 + 2\partial_{01}^2 v_1 + 2\partial_{02}^2 v_2 + 2\partial_{03}^2 v_3 = u_0, \\
\partial_0^2 v_1 + \partial_1^2 v_1 + \partial_2^2 v_1 + \partial_3^2 v_1 = u_1, \\
\partial_0^2 v_2 + \partial_1^2 v_2 + \partial_2^2 v_2 + \partial_3^2 v_2 = u_2, \\
\partial_0^2 v_3 + \partial_1^2 v_3 + \partial_2^2 v_3 + \partial_3^2 v_3 = u_3.
\end{cases} \tag{15}
$$

The above system is decoupled and hence can be solved with the boundary condition $v|_{\partial\Omega} = 0$. Then we use (12) to solve for λ. This completes the proof of the theorem for $n = 3$.

Proof for $n \geq 4$

From (14), we have

$$\begin{bmatrix} \Delta v_0 + \alpha \partial^2_{0j} v_j \\ \Delta v_1 + \beta \partial^2_{1j} v_j \\ \Delta v_2 + \beta \partial^2_{0j} v_j \\ \vdots \\ \Delta v_n + \beta \partial^2_{nj} v_j \end{bmatrix} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}; \quad v|_{\partial\Omega} = 0. \tag{16}$$

Our aim is to show that the coupled system (16) is uniquely solvable.

Proof for $n \geq 4$

To prove this it is enough (Taylor's book on PDE 1) to show that it is strongly elliptic with zero kernel and zero co-kernel. Let $u := (u_0, u_1, u_2, \cdots, u_n)$ and define the $(1+n) \times (1+n)$ matrix $A(x, \partial)$ by

$$A(x;\partial) := \begin{bmatrix} \Delta + \alpha\partial_0^2 & \alpha\partial_{01}^2 & \alpha\partial_{02} & \cdots & \alpha\partial_{0n}^2 \\ \beta\partial_{10}^2 & \Delta + \beta\partial_1^2 & \beta\partial_{12}^2 & \cdots & \beta\partial_{1n}^2 \\ \beta\partial_{20}^2 & \beta\partial_{21}^2 & \Delta + \beta\partial_2^2 & \cdots & \beta\partial_{2n}^2 \\ \beta\partial_{30}^2 & \beta\partial_{31}^2 & \beta\partial_{32}^2 & \cdots & \beta\partial_{3n}^2 \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \vdots & \vdots & \vdots & \vdots & \dots \\ \beta\partial_{n0}^2 & \beta\partial_{n1}^2 & \beta\partial_{n2}^2 & \cdots & \Delta + \beta\partial_n^2 \end{bmatrix}.$$

Proof for $n \geq 4$

With this (16) becomes

$$\begin{cases} A(x;\partial)v = u, \text{ in } \Omega \\ v = 0, \text{ on } \partial\Omega \end{cases} \tag{17}$$

Strong ellipticity for $A(x;\partial)$

Symbol $A(x;\xi)$ for operator $A(x;\partial)$ is given by

$$A(x;\xi)=\begin{bmatrix} |\xi|^2+\alpha\xi_0^2 & \alpha\xi_0\xi_1 & \alpha\xi_0\xi_2 & \cdots & \alpha\xi_0\xi_n \\ \beta\xi_1\xi_0 & |\xi|^2+\beta\xi_1^2 & \beta\xi_1\xi_2 & \cdots & \beta\xi_1\xi_n \\ \beta\xi_2\xi_0 & \beta\xi_2\xi_1 & |\xi|^2+\beta\xi_2^2 & \cdots & \beta\xi_2\xi_n \\ \beta\xi_3\xi_0 & \beta\xi_3\xi_1 & \beta\xi_3\xi_2 & \cdots & \beta\xi_3\xi_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \beta\xi_n\xi_0 & \beta\xi_n\xi_1 & \beta\xi_n\xi_2 & \cdots & |\xi|^2+\beta\xi_n^2 \end{bmatrix}. \tag{18}$$

In order to prove the strong ellipticity for $A(x;\xi)$ enough to prove that the matrix $P(x;\xi):=\frac{A(x;\xi)+A^T(x;\xi)}{2}$ is positive definite.

Positivity for $P(x;\xi)$

Let $\eta, \xi \in \mathbb{R}^{1+n} \setminus \{0\}$ be two column vectors, then we consider

$$\begin{aligned}
\eta^T P(x;\xi)\eta &= |\xi|^2|\eta|^2 + (\alpha - 1)\,\xi_0^2\eta_0^2 + \xi_0\eta_0\,(\xi \cdot \eta) \\
&\quad + (1-\beta)\xi_0\eta_0\,(\xi\cdot\eta - \xi_0\eta_0) + \beta\xi\cdot\eta\,(\xi\cdot\eta - \xi_0\eta_0) \\
&= |\xi|^2|\eta|^2 + (\alpha+\beta-2)\,\xi_0^2\eta_0^2 + \beta\,(\xi\cdot\eta)^2 \\
&\quad + 2\,(1-\beta)\,(\xi\cdot\eta)\,\xi_0\eta_0.
\end{aligned} \tag{19}$$

Substituting the values of α and β, we have

$$\eta^T P(x;\xi)\eta = \frac{|\xi|^2|\eta|^2}{n-1}\left(n - 1 + (n-3)\left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)^2 + 4\left(\frac{(\xi_0\eta_0)\,(\xi\cdot\eta)}{|\xi|^2|\eta|^2}\right)\right)$$

Positivity for $P(x;\xi)$

Define $a := \frac{\xi_0\eta_0}{|\xi||\eta|}$ and $b = \frac{\xi'\cdot\eta'}{|\xi||\eta|}$, then clearly $|a| \leq 1$, $|b| \leq 1$ and $|a+b| \leq 1$. Using these in above equation, we get

$$
\begin{aligned}
\eta^T P(x;\xi)\eta &= \frac{|\xi|^2|\eta|^2}{n-1}\Big(n-1+(n-3)\,(a+b)^2+4a\,(a+b)\Big)\\
&= \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+(n+1)\,a^2+2(n-1)ab+(n-3)b^2\right)\\
&\geq \frac{|\xi|^2|\eta|^2}{n-1}\Big(n-1+(n+1)a^2-(n-1)\left(a^2+b^2\right)+(n-3)b^2\Big)\\
&\geq \frac{|\xi|^2|\eta|^2}{n-1}\left(n-1+2a^2-2b^2\right) \geq \frac{n-3}{n-1}|\xi|^2|\eta|^2\\
&\geq C|\xi|^2|\eta|^2;\ \text{for some constant } C>0 \text{ provided } n \geq 4.
\end{aligned}
$$

This prove the strong ellipticity for $A(x;\partial)$.

Kernel of $A(x;\partial)$

Here we have to prove that the following BVP

$$\begin{cases} A(x;\partial)v = 0, \text{ in } \Omega \\ v = 0, \text{ on } \partial\Omega \end{cases}$$

has only zero solution. Substituting the expression for $A(x;\partial)$ and v, we have

$$\Delta v_0 + \left(1 + \frac{2}{n-1}\right)\sum_{k=0}^{n} \partial^2_{0k} v_k = 0, \;\; v|_{\partial\Omega} = 0, \tag{20}$$

$$\Delta v_j + \left(1 - \frac{2}{n-1}\right)\sum_{k=0}^{n} \partial^2_{jk} v_k = 0, \;\; v|_{\partial\Omega} = 0, \;\; 1 \le j \le n. \tag{21}$$

Kernel of $A(x;\partial)$

Multiplying (20) by v_0 and (21) by v_j and integrating over Ω, we have

$$\int\limits_{\Omega} |\nabla v_0(x)|^2 \mathrm{d}x + \left(1 + \frac{2}{n-1}\right) \int\limits_{\Omega} \nabla \cdot v(x) \partial_0 v_0(x) \mathrm{d}x = 0 \quad (22)$$

and

$$\int\limits_{\Omega} |\nabla v_j(x)|^2 \mathrm{d}x + \left(1 - \frac{2}{n-1}\right) \int\limits_{\Omega} \nabla \cdot v(x) \partial_j v_j(x) \mathrm{d}x = 0. \quad (23)$$

Adding the set of equations in (22) and (23), we get

$$\int\limits_{\Omega} \sum_{j=0}^{n} |\nabla v_j|^2 \mathrm{d}x + \beta \int\limits_{\Omega} |\nabla \cdot v|^2 \mathrm{d}x + \frac{4}{n-1} \int\limits_{\Omega} \nabla \cdot v \left(\partial_0 v_0\right) \mathrm{d}x = 0. \quad (24)$$

Kernel of $A(x;\partial)$

$a := \partial_0 v_0$, $b := \sum_{j=1}^{n} \partial_j v_j$ and $c := \sum_{j=0}^{n} |\nabla v_j|^2 - |\partial_0 v_0|^2$. Using these in (24), we have

$$\int_{\Omega} \left(c + a^2 + \frac{n-3}{n-1} (a+b)^2 + \frac{4}{n-1} \left(a^2 + ab\right) \right) \mathrm{d}x = 0.$$

Now after combining the similar terms, we get

$$\int_{\Omega} \left(2na^2 + 2\,(n-1)\,ab + (n-3)\,b^2 + (n-1)\,c\right) \mathrm{d}x = 0.$$

Now lets view the integrand in the above equation as a quadratic in a and then its discriminant $D_n(x)$ given by

$$\begin{aligned} D_n(x) &= 4(n-1)^2 b^2 - 8n\left((n-3)b^2 + (n-1)c\right) \\ &= 4\left[\left(-n^2 + 4n + 1\right) b^2 - 2n(n-1)c\right]. \end{aligned}$$

Kernel of $A(x;\partial)$

Using the fact that $nc \geq b^2$, we get

$$D_n(x) \leq 4\left(-n^2 + 2n + 3\right) b^2 < 0; \text{ if } b^2 \neq 0 \text{ and } n \geq 4$$

but if $D_n(x) < 0$ then we have the integrand in (24) is strictly positive which cannot be true since integration in (24) is zero. Hence we have $b = 0$, using this in (24), we get $v \equiv 0$ and hence $\mathrm{Ker} A(x;\partial) = \{0\}$.

- Similar arguments show that $\text{Co-ker} A(x;\partial) = \{0\}$.

This completes the proof for existence of v in $n \geq 4$.

Existence of λ and $\widetilde{F}$

Using v in (12) i.e. in

$$\operatorname{trace}(F) = (n-1)\lambda + \delta v$$

we get

$$\lambda = \frac{\operatorname{trace}(F) - \delta v}{n-1}$$

and then take

$$\widetilde{F} = F - \lambda g - \mathrm{d}v$$

we get the decomposition formula for F.

Thank you very much for your attention