Modern Challenges in Imaging

Generalized Radon Transforms and Applications

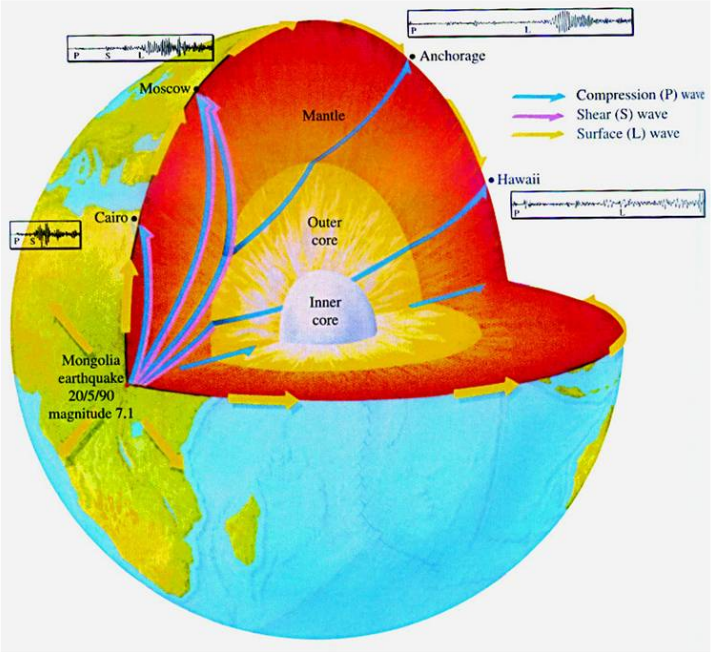
Gunther Uhlmann

University of Washington and HKUST

Tufts University, Boston, August 5, 2019

Travel Time Tomography (Transmission)

Global Seismology



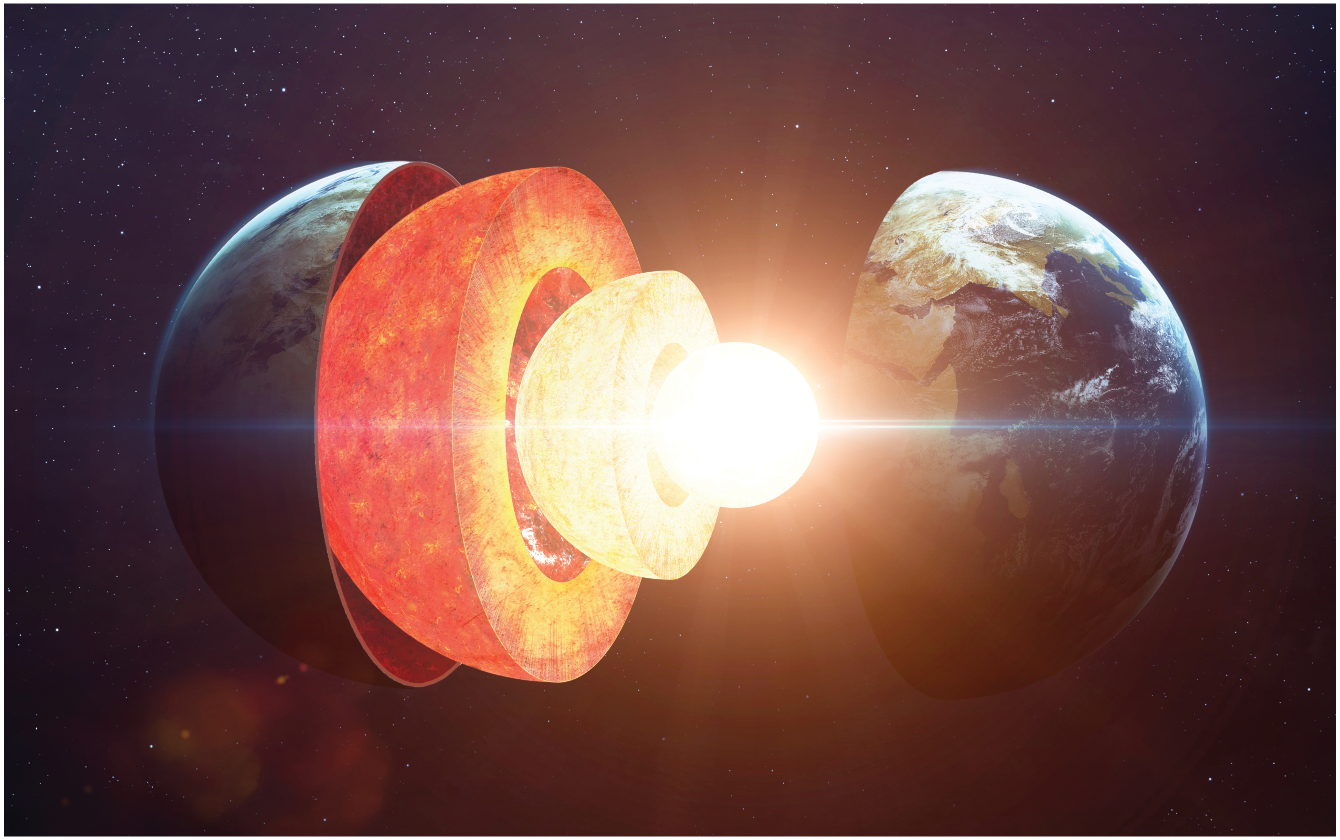
Inverse Problem: Determine inner structure of Earth by measuring travel time of seismic waves.

Seismic Waves

(Loading seismic waves.mpg)

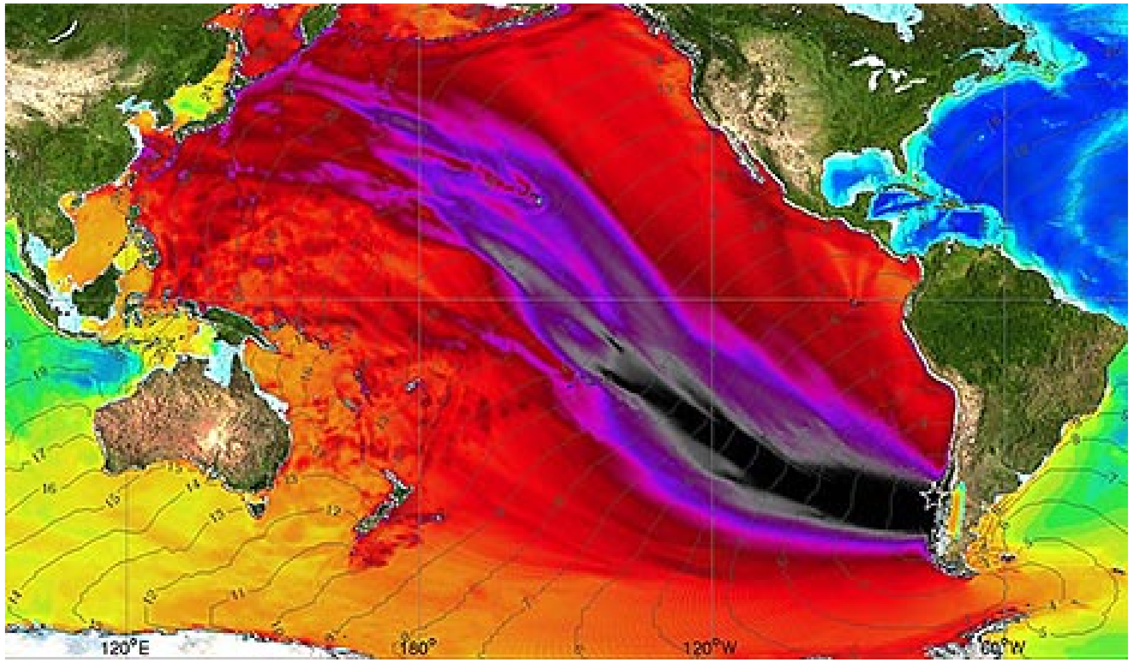
Travel Time Tomography

Long-awaited mathematics proof could help scan Earth's innards



Nature, Feb, 2017

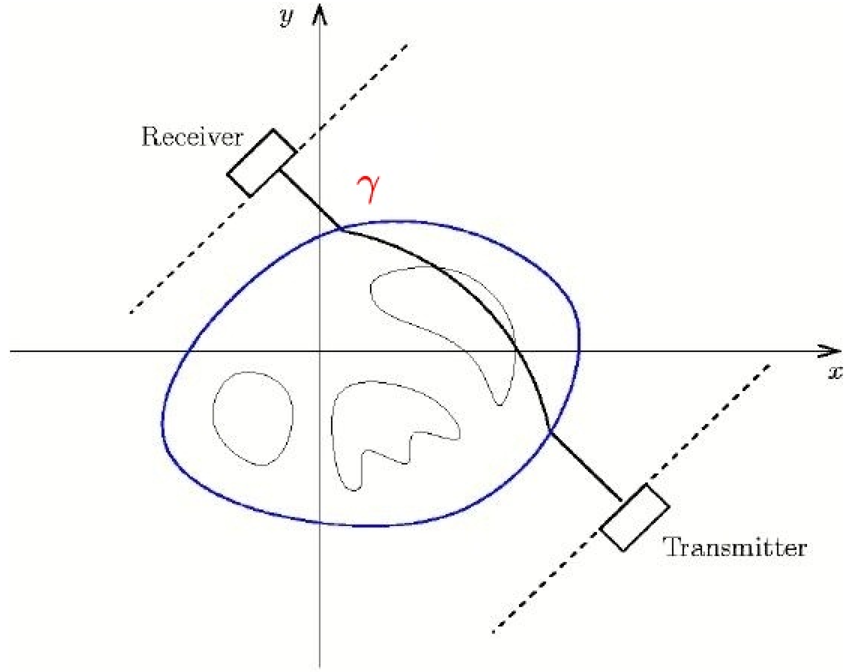
Tsunami of 1960 Chilean Earthquake



Black represents the largest waves, decreasing in height through purple, dark red, orange and on down to yellow. In 1960 a tongue of massive waves spread across the Pacific, with big ones throughout the region.

Human Body Seismology

ULTRASOUND TRANSMISSION TOMOGRAPHY(UTT)

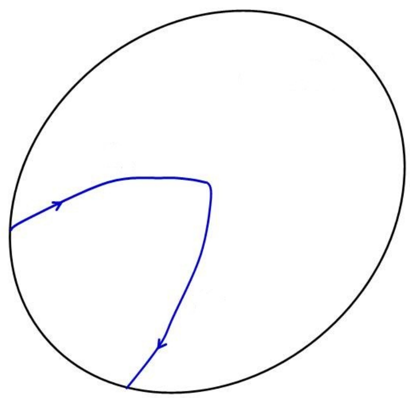


$$T = \int_\gamma \frac{1}{c(x)}\,ds \quad = \text{ Travel Time (Time of Flight).}$$

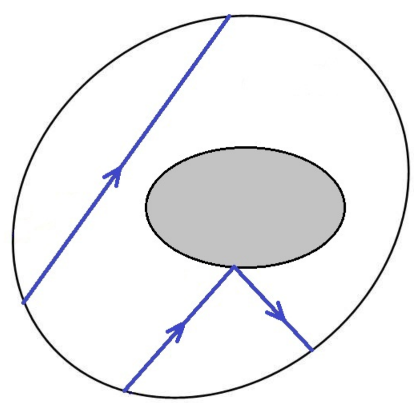
REFLECTION TOMOGRAPHY

Scattering

Points in medium

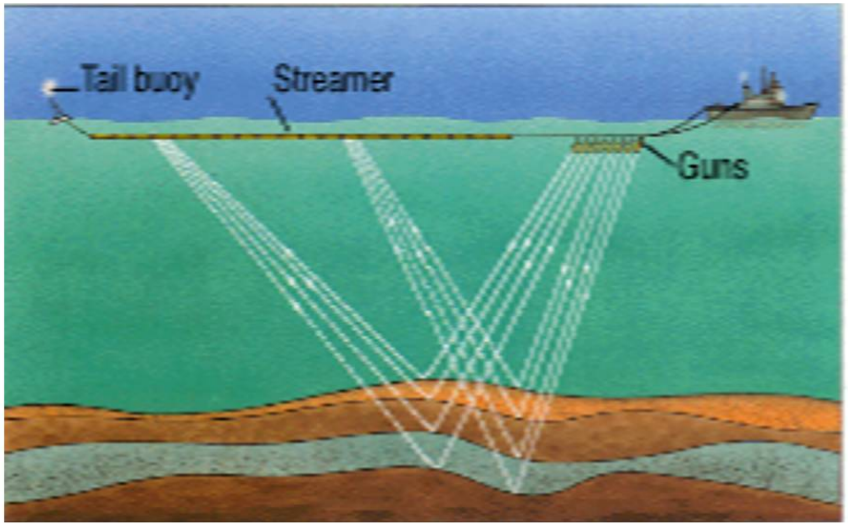


Obstacle



REFLECTION TOMOGRAPHY

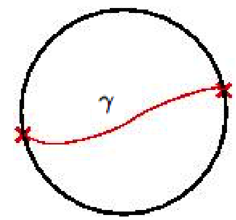
Oil Exploration



Ultrasound

TRAVELTIME TOMOGRAPHY (Transmission)

Motivation:Determine inner structure of Earth by measuring travel times of seismic waves



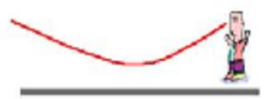
Herglotz (1905), Wiechert-Zoeppritz (1907)

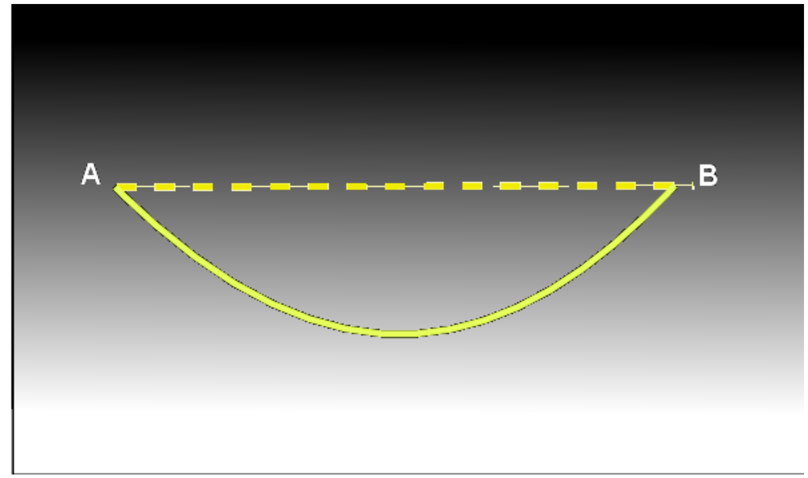
Sound speed $c(r)$, $r = |x|$

$$\frac{d}{dr}\left(\frac{r}{c(r)}\right) > 0$$

$T = \int_\gamma \frac{1}{c(r)}$. What are the curves of propagation γ?

Ray Theory of Light: Fermat's principle



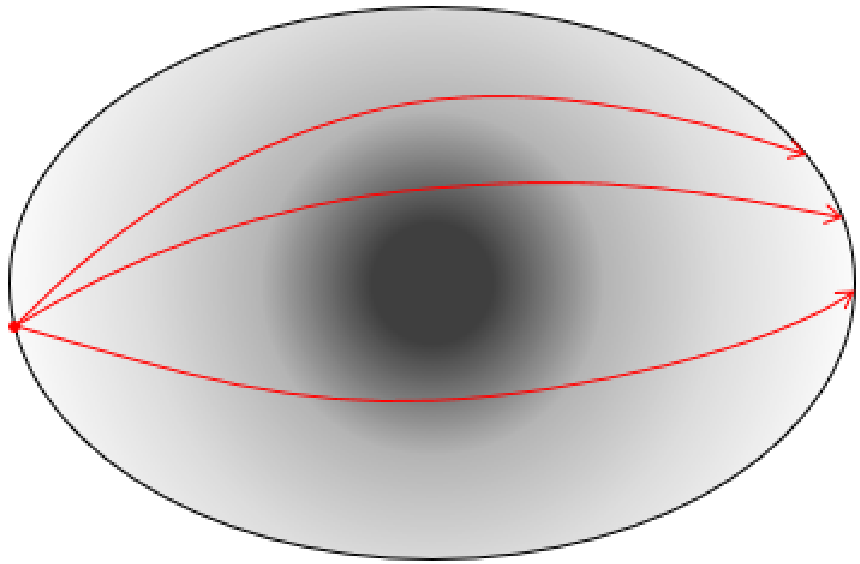


Fermat's principle. Light takes the shortest optical path from A to B (solid line) which is not a straight line (dotted line) in general. The optical path length is measured in terms of the refractive index n integrated along the trajectory. The greylevel of the background indicates the refractive index; darker tones correspond to higher refractive indices.

Geodesics (Rays)

The curves are geodesics of a metric $ds^2 = \frac{1}{c^2(r)}dx^2$, or more generally, $ds^2 = \frac{1}{c^2(x)}dx^2$. Velocity $v(x,\xi) = c(x)$, $\quad |\xi| = 1$ (isotropic)

Geodesics minimize length (time) locally, $\frac{ds}{c}$.



Geodesics in a medium with a slow region in the center

Geodesics in Phase Space

Hamiltonian is given by

$$H_c(x,\xi) = \frac{1}{2}\Big(c^2(x)|\xi|^2 - 1\Big)$$

$X_c(s, X^0) = \Big(x_c(s, X^0), \xi_c(s, X^0)\Big)$ be bicharacteristics,

sol. of $$\frac{dx}{ds} = \frac{\partial H_c}{\partial \xi}, \qquad \frac{d\xi}{ds} = -\frac{\partial H_c}{\partial x}$$

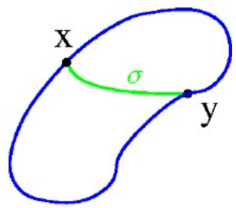
$x(0) = x^0$, $\xi(0) = \xi^0$, $X^0 = (x^0, \xi^0)$, where $\xi^0 \in \mathcal{S}_c^{n-1}(x^0)$

$$\mathcal{S}_c^{n-1}(x) = \Big\{\xi \in \mathbb{R}^n;\ H_c(x,\xi) = 0\Big\}.$$

Geodesics Projections in x: $x(s)$.

Boundary distance function

The travel time information is encoded in the boundary distance function



$x, y \in \partial M$

$$d_c(x,y) = \inf_{\substack{\sigma(0)=x \\ \sigma(1)=y}} L(\sigma)$$

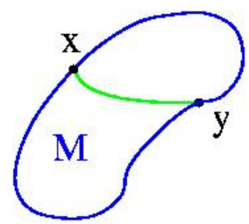
$L(\sigma) =$ length of curve σ

$$L(\sigma) = \int_0^1 \frac{1}{c} \left|\frac{d\sigma}{dt}\right| dt$$

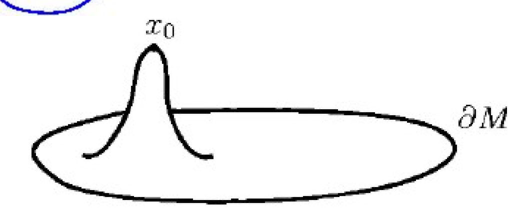
Inverse problem

Determine $\boxed{c}$ knowing $\boxed{d_c(x,y)}$ $x, y \in \partial M$

Obstructions



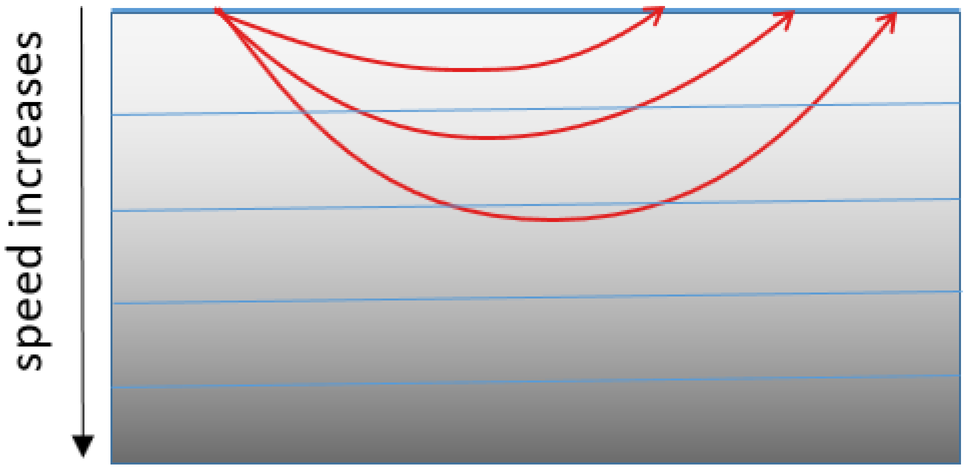
$dc \Rightarrow c$?



$$d_c(x_0, \partial M) > \sup_{x,y \in \partial M} d_c(x, y)$$

Need an a-priori condition to recover c from dc.

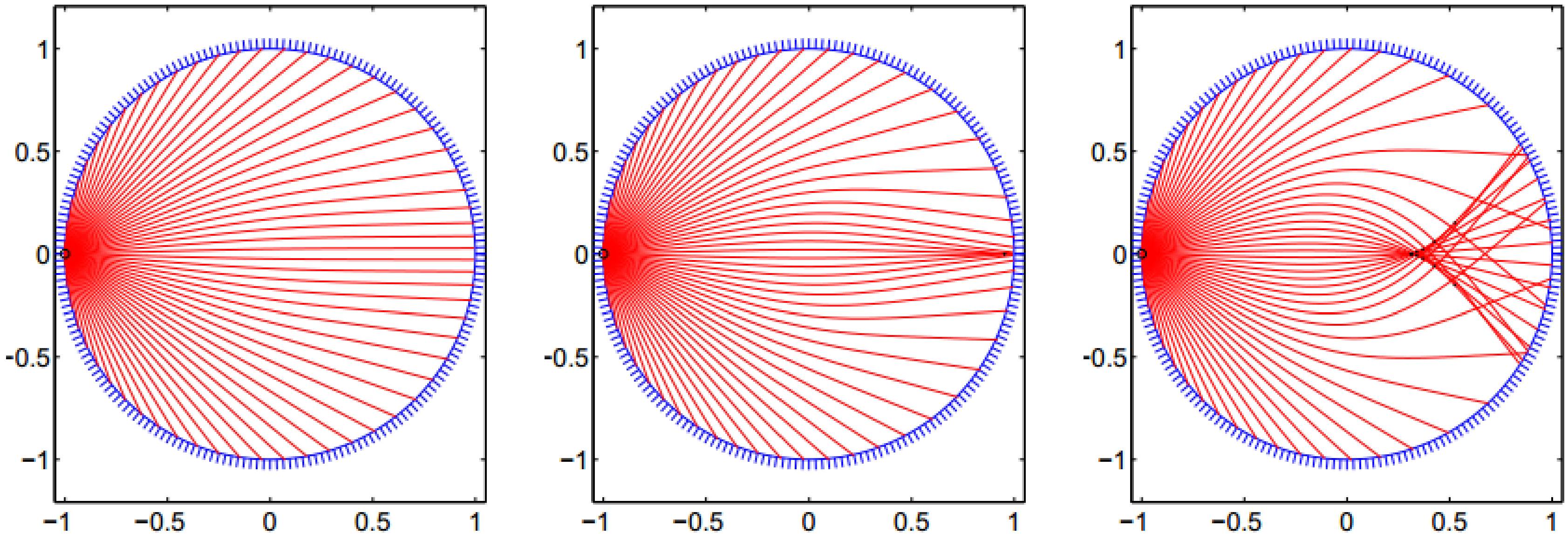
DEF (M, c) is simple if given two points $x, y \in \partial M$, $\exists!$ minimizing geodesic joining x and y and ∂M is strictly convex



strictly convex

THEOREM(Mukhometov, 1975) One can determine c uniquely and stably from dc if (M, c) is simple.

Speeds Satisfying the Herglotz condition



$k = 0.20$ (simple) $\quad k = 0.49$ (non-simple) $\quad k = 1.23$ (non-simple)

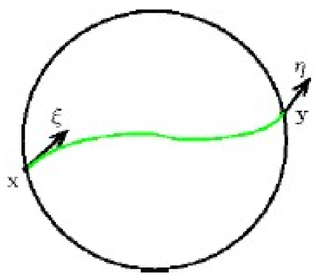
$$c_k(r) = exp(k\, exp(-\frac{r^2}{2\sigma^2})),\ 0 \leq \sigma \leq 1,\ \sigma \text{ fixed}$$

Francois Monard: SIAM J. Imaging Sciences (2014)

Scattering Relation

d_c only measures first arrival times of waves.

We need to look at behavior of all geodesics



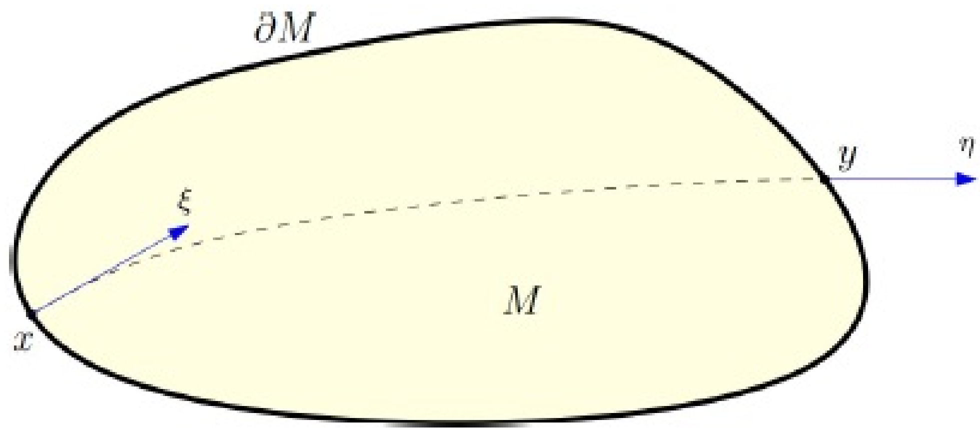
$$\|\xi\|_c = \|\eta\|_c = 1$$

$\alpha_c(x, \xi) = (y, \eta)$, α_c is SCATTERING RELATION

If we know direction and point of entrance of geodesic then we know its direction and point of exit.

Travel Time Tomography

Define the scattering relation α_c.



$$\alpha_c : (x, \xi) \to (y, \eta).$$

α_c, d_c follows all geodesics.

Inverse Problem: *Do α_c, d_c determine c?*

Non-simple Speeds

IP: Do α_g, d_c determine c?

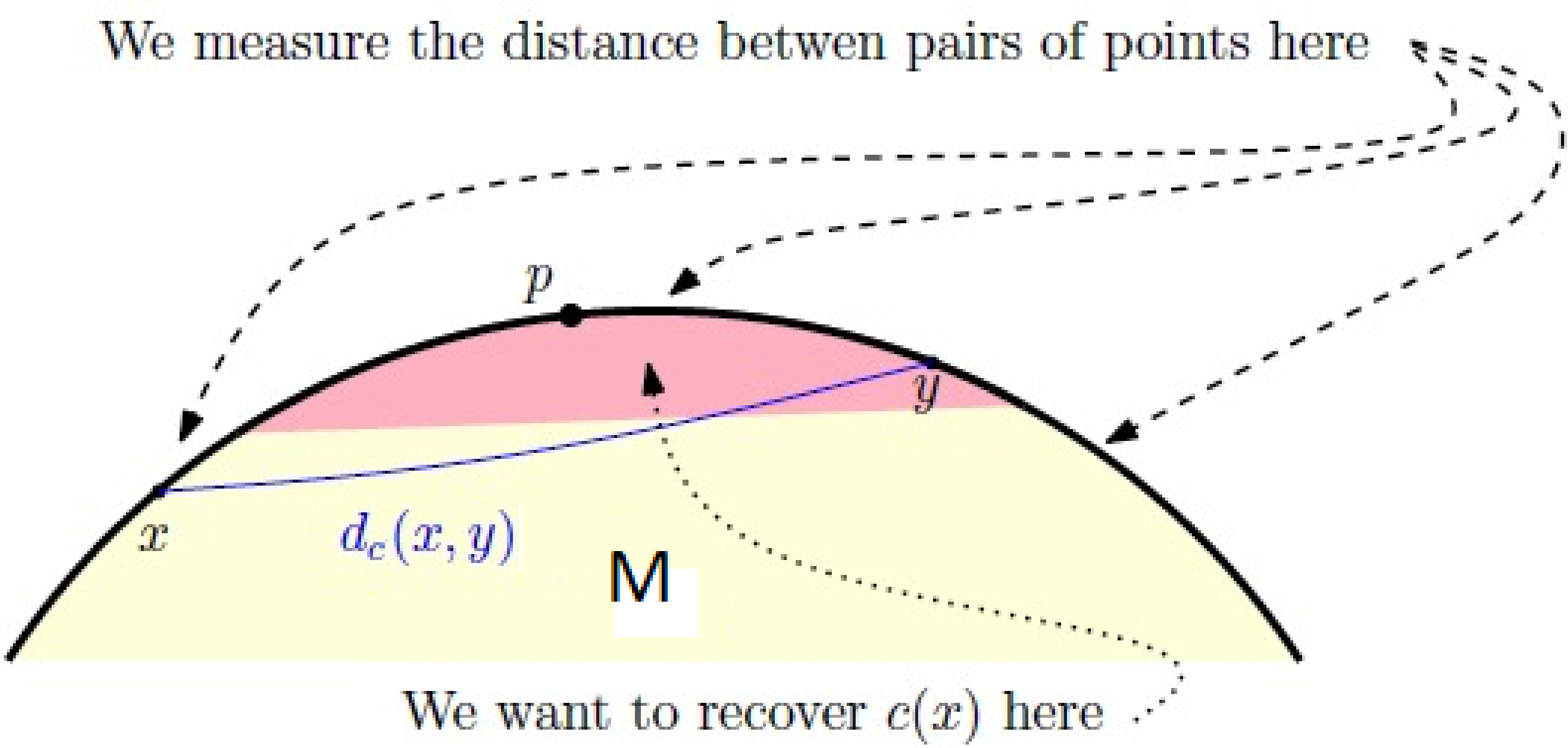
Remark: If (M, c) is simple, α_c is equivalent to d_c.

For non-simple metrics (caustics and/or non-convex boundary), this is the right problem to study.

Some results: local generic rigidity near a class of non-simple sound speeds (Stefanov-U, 2009), real-analytic sound speeds satisfying a mild condition (Vargo, 2010), stability estimates for a class of non-simple sound speeds (Bao-H. Zhang 2014, 2017), foliation condition (Stefanov-U-Vasy, 2016, 2017).

Partial Data

Travel time with partial data: Does d_c, known on $\partial M \times \partial M$ near some p, determine c near p uniquely?



Result on Partial Data

Theorem (Stefanov-U-Vasy, 2016). Let $\dim M \geq 3$. If ∂M is strictly convex near p for c and $\tilde{c}$, and $d_c = d_{\tilde{c}}$ near (p, p), then $c = \tilde{c}$ near p.

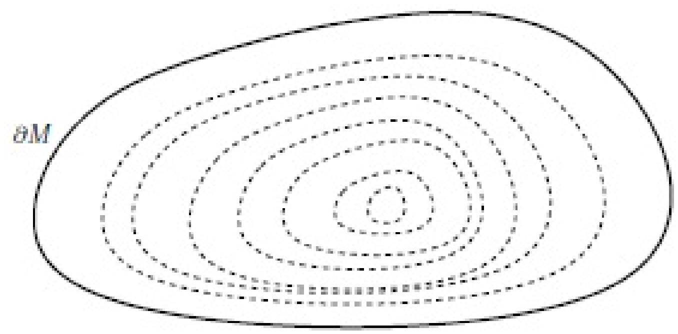
Also stability and reconstruction.

The only results so far of similar nature is for real analytic sound speeds (Lassas-Sharafutdinov-U, 2003). We can recover the whole jet of the metric at ∂M and then use analytic continuation.

Foliation condition

We could use a layer stripping argument to get deeper and deeper in M and prove that one can determine c in the whole M.

Foliation condition: M is foliated by strictly convex hypersurfaces if, up to a nowhere dense set, $M = \cup_{t\in[0,T)}\Sigma_t$, where Σ_t is a smooth family of strictly convex hypersurfaces and $\Sigma_0 = \partial M$.



A more general condition: several families, starting from outside M.

Global result isotropic case

Theorem (Stefanov-U-Vasy, 2016)**.** Let $\dim M \geq 3$, let c and $\widetilde{c}$ be two smooth sound speeds on M, let ∂M be strictly convex with respect to both c and $\widetilde{c}$. Assume that M can be foliated by strictly convex hypersurfaces for c. Then if $\alpha_c = \alpha_{\widetilde{c}}, d_c = d_{\widetilde{c}}$ we have $c = \widetilde{c}$ in M.

Also stability and reconstruction.

Examples: The foliation condition is satisfied for strictly convex domains of non-negative sectional curvature, simply connected domains with non-positive sectional curvature and simply connected domains with no focal points.

Foliation condition is an analog of the Herglotz, Wieckert-Zoeppritz condition for non radial speeds.

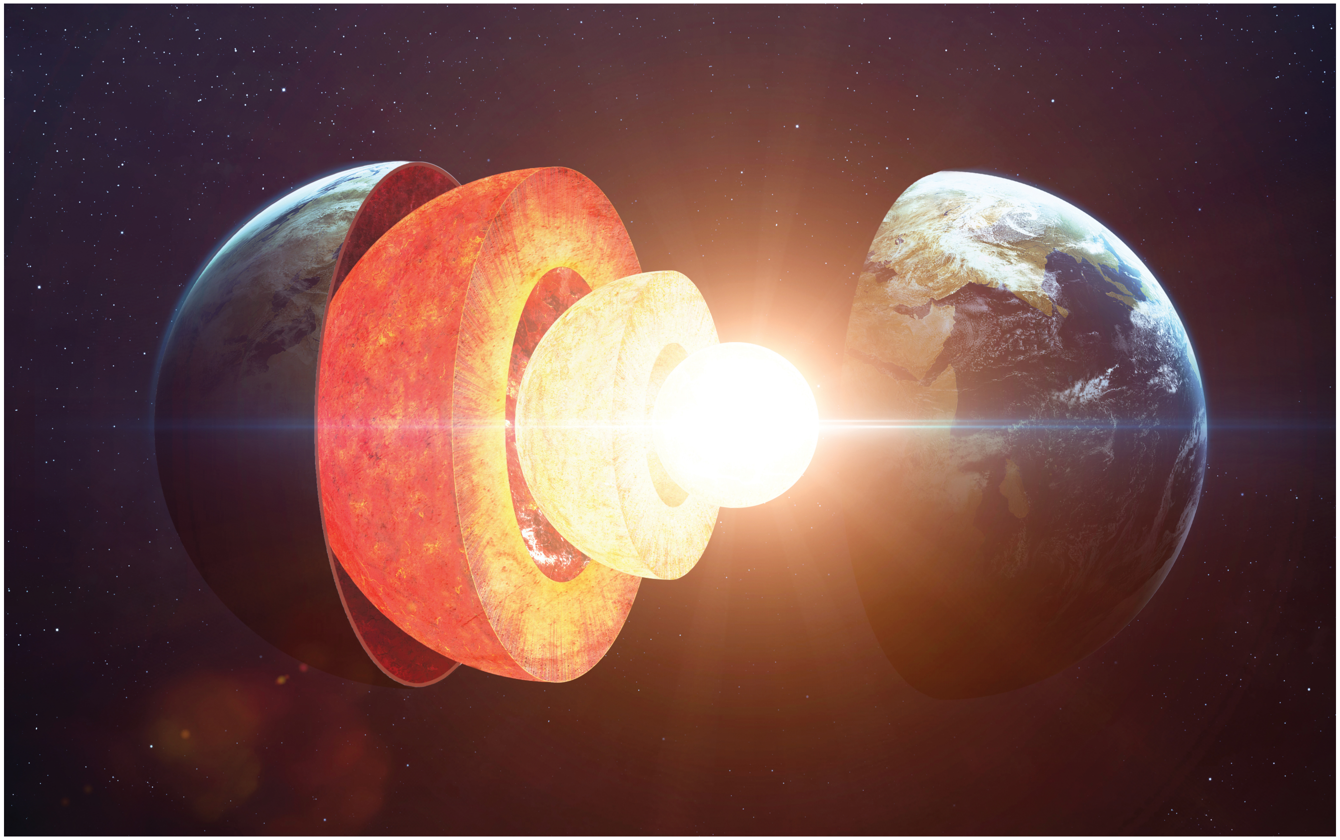
Example: Herglotz and Wiechert & Zoeppritz showed that one can determine a radial speed $c(r)$ in the ball $B(0,1)$ satisfying

$$\boxed{\frac{d}{dr}\frac{r}{c(r)} > 0.}$$

The uniqueness is in the class of radial speeds.

One can check directly that their condition is equivalent to the following one: the Euclidean spheres $\{|x| = t\}$, $t \leq 1$ are strictly convex for $c^{-2}dx^2$ as well. Then $B(0,1)$ satisfies the foliation condition. Therefore, if $\tilde{c}(x)$ is another speed, not necessarily radial, with the same distance function and scattering relation, equal to c on the boundary, then $c = \tilde{c}$. There could be conjugate points. Also we have stability and reconstruction.

Long-awaited mathematics proof could help scan Earth's innards



Nature, Feb, 2017

Ideas of the proof in isotropic case

The proof is based on two main ideas.

First, we use the approach in a recent paper by U-Vasy (2016) on the linearized problem with partial data.

Second, we convert the non-linear boundary rigidity problem to a "pseudo-linear" one. Straightforward linearization, which works for the problem with full data, fails here.

Linearized Problem

Let c be a sound speed. Linearizing $c \mapsto d_c$ leads to the *ray transform*

$$If(x,\xi) = \int_0^{\tau(x,\xi)} f(\gamma(t,x,\xi))\,dt$$

where $x \in \partial M$ and $\xi \in S_x M = \{\xi \in T_x M\,;\,|\xi| = 1\}$.

Here $\gamma(t,x,\xi)$ is the geodesic starting from point x in direction ξ, and $\tau(x,\xi)$ is the time when γ exits M. We assume that (M,c) is *nontrapping*, i.e. τ is always finite.

Inversion of X-ray Transform (Radon 1917)

- $If(x,\theta) = \int f(x + t\theta)dt, \quad |\theta| = 1$

- $(-\Delta)^{1/2} I^* I f = cf, \quad c \neq 0$

- $(-\Delta)^{-1/2} f = \int \frac{f(y)}{|x-y|^{n-1}} dy$

I^*I is an elliptic pseudodifferential operator of order -1.

Linearized Problem with Partial Data

U-Vasy result: Consider the inversion of the geodesic ray transform

$$If(\gamma) = \int f(\gamma(s))\,ds$$

known for geodesics intersecting some neighborhood of $p \in \partial M$ (where ∂M is strictly convex) "almost tangentially". It is proven that those integrals determine f near p uniquely. It is a Helgason support type of theorem for non-analytic curves! This was extended recently by H. Zhou for arbitrary curves (∂M must be strictly convex w.r.t. them) and non-vanishing weights.

The main idea in U-Vasy is the following:

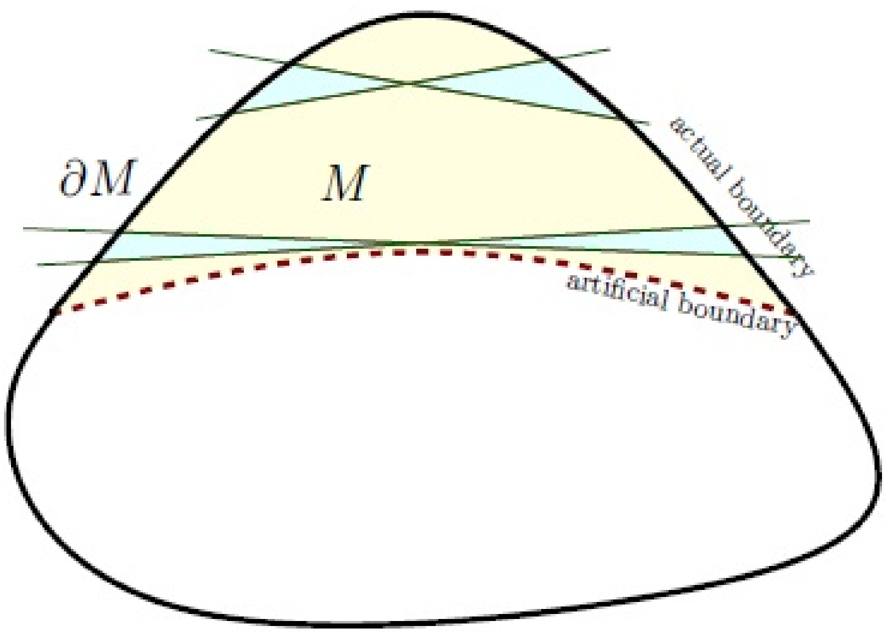
Introduce an artificial, still strictly convex boundary near p which cuts a small subdomain near p. Then use Melrose's scattering calculus to show that the I, composed with a suitable "back-projection" is elliptic in that calculus. Since the subdomain is small, it would be invertible as well.

U-Vasy

Consider

$$Pf(z) := I^*\chi If(z) = \int_{S_zM} x^{-2}\chi If(\gamma_{z,v})dv,$$

where χ is a smooth cutoff sketched below (angle $\sim x$), and x is the distance to the artificial boundary.



Inversion of local geodesic transform

$$Pf(z) := I^* \chi I f(z) = \int_{S_z M} x^{-2} \chi I f(\gamma_{z,v}) dv,$$

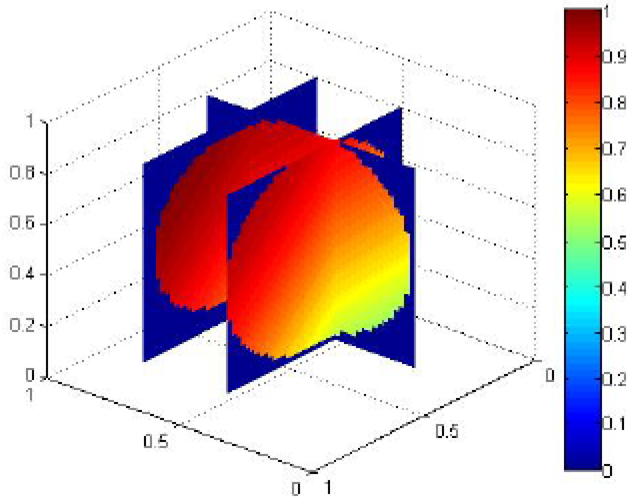
Main result: P is an elliptic pseudodifferential operator in Melrose's scattering calculus.

There exists A such that $AP = Identity + R$

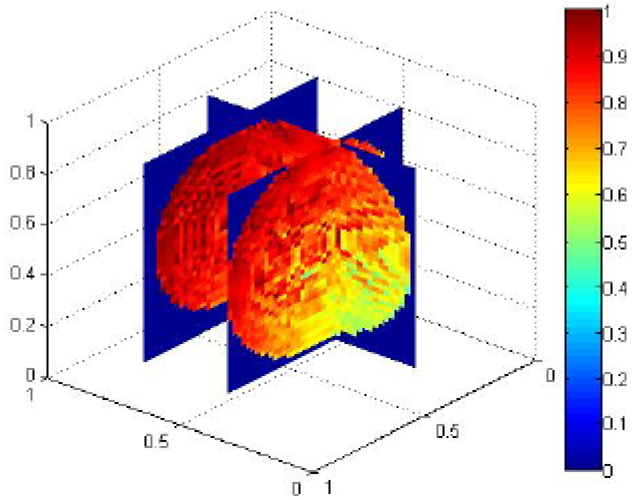
This is Fredholm and R has a small norm in a neighborhood of p. Therefore invertible near p using Neumann series.

$$\begin{aligned} f =& (Identity + R)^{-1} AP \\ =& \sum_{j=0}^{\infty} K^j f, \quad \|K\| < 1 \end{aligned}$$

Some numerical results for inverse geodesic X-ray transform



(a) exact solution for f_1

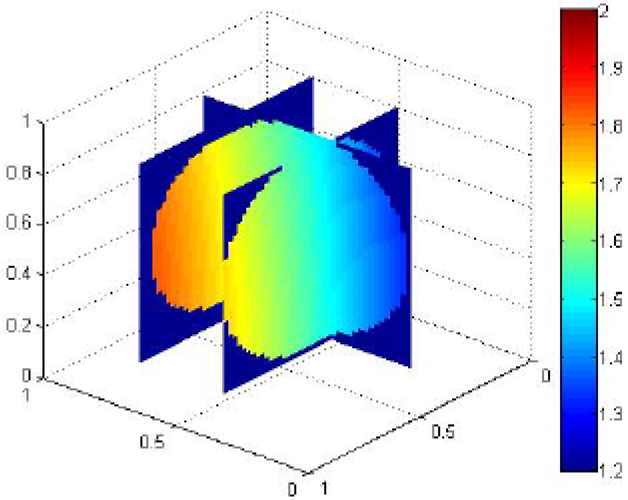


(b) approximate solution for f_1

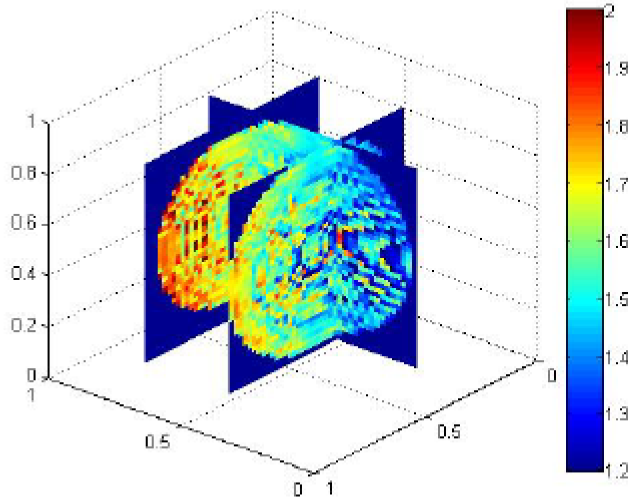
$$f_1 = 0.01 + \sin\left(2\pi(x+y+z)/10\right)$$

(T.-S. Au, E. Chung - U, 2019)

Some numerical results for inverse geodesic X-ray transform



(c) exact solution for f_2

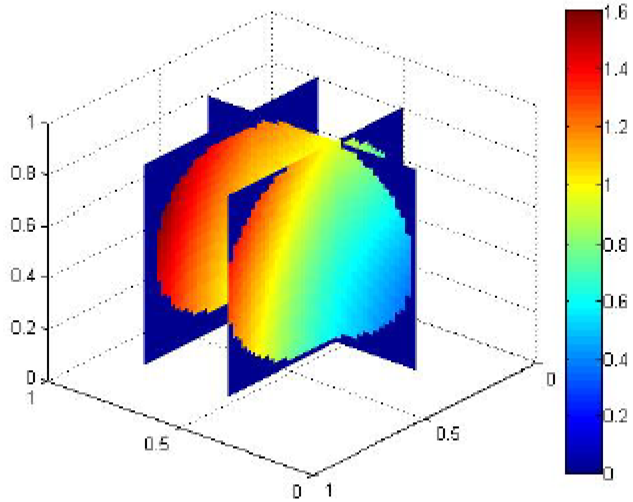


(d) approximate solution for f_2

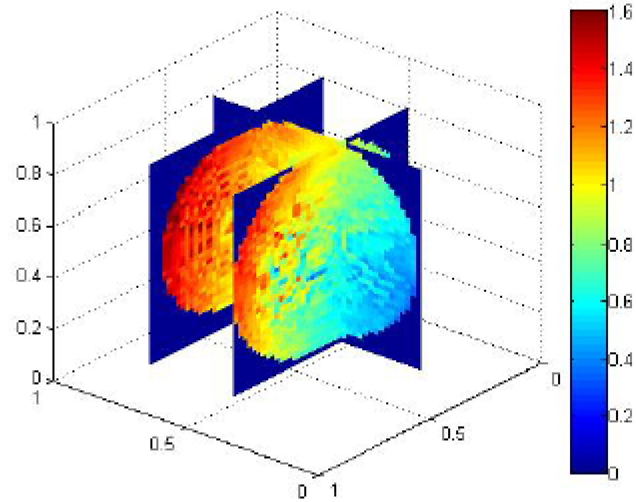
$$f_2 = 0.01 + \sin\Big(2\pi(x+y)/10\Big) + \cos\Big(2\pi z/20\Big)$$

(T.-S. Au, E. Chung - U, 2019)

Some numerical results for inverse geodesic X-ray transform



(e) exact solution for f_3

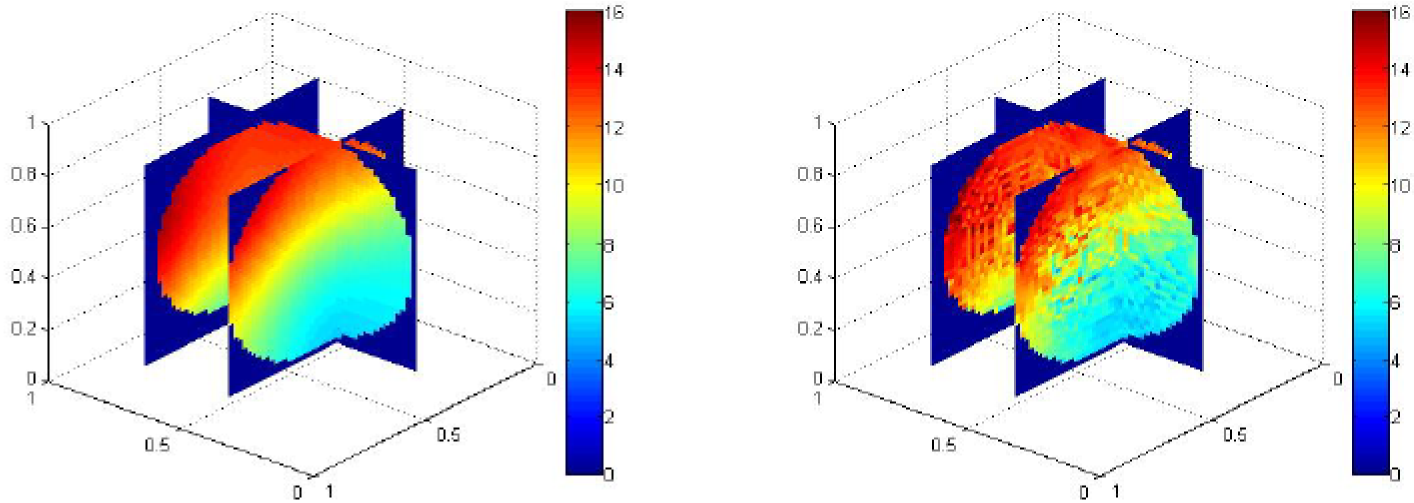


(f) approximate solution for f_3

$$f_3 = x + y^2 + z^2/2$$

(T.-S. Au, E. Chung - U, 2019)

Some numerical results for inverse geodesic X-ray transform

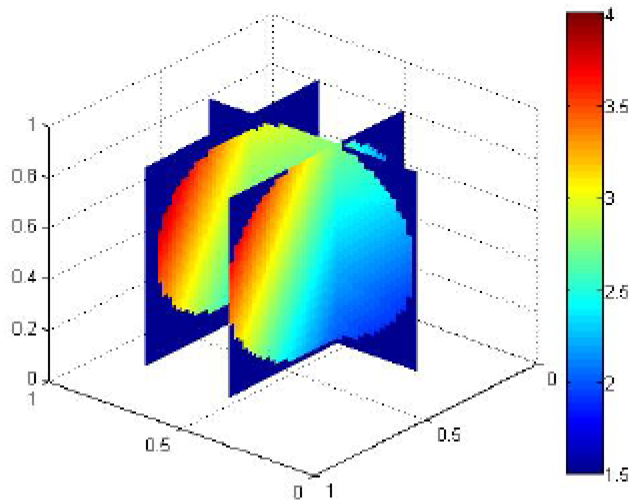


(a) exact solution for f_4 (b) approximate solution for f_4

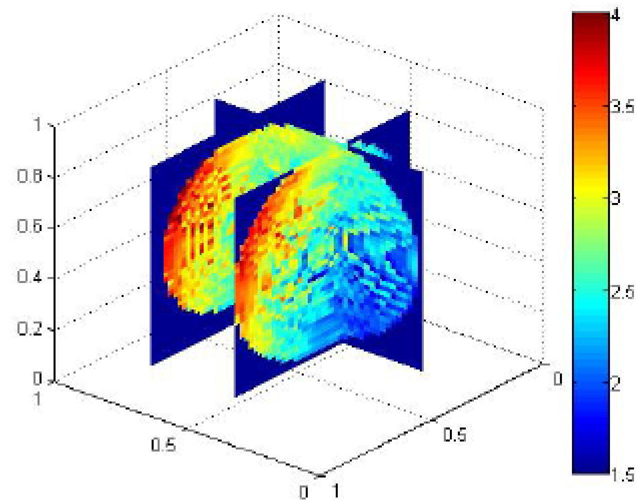
$$f_4 = 1 + 6x + 4y + 9z + \sin\left(2\pi(x+z)\right) + \cos\left(2\pi y\right)$$

(T.-S. Au, E. Chung - U, 2019)

Some numerical results for inverse geodesic X-ray transform



(c) exact solution for f_5



(d) approximate solution for f_5

$$f_5 = x + e^{y+z/2}$$

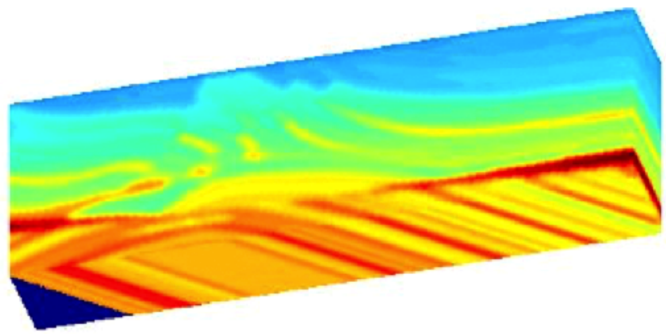
(T.-S. Au, E. Chung - U, 2019)

- Relative errors for using up to 4 terms in the Neumann series

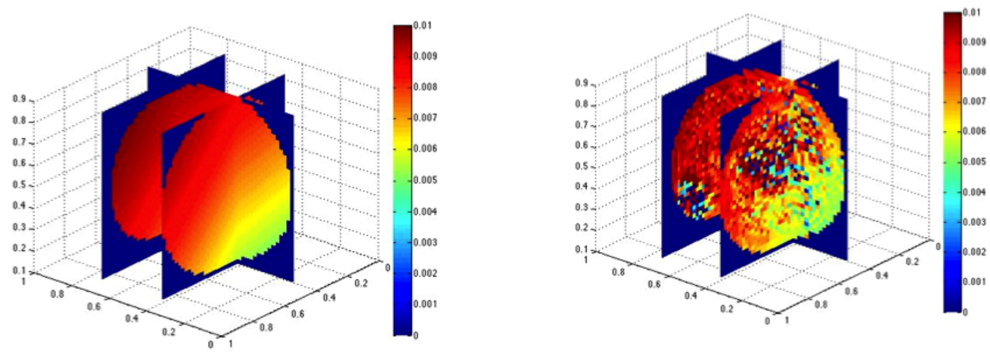
relative error	f_1	f_2	f_3	f_4	f_5
n=0	37.1%	37.08%	37.13%	37.27%	37.25%
n=1	15.74 %	15.63%	15.81%	16.2%	16.32 %
n=2	8.92%	8.65%	9.09%	9.98%	10.28%
n=3	6.99%	6.55%	7.26%	8.61%	9.02%

Nonlinear Problem

- We test the method using a spherical section of the Marmousi model



- Results



	$n = 0$	$n = 1$	$n = 2$	$n = 3$
relative error	40.92%	19.89%	14.48%	14.20%
relative error with 5% noisy data	42.15%	22.33%	17.47%	17.12%

Elasticity

The isotropic elastic equation is given by

$$(\partial_t^2 - E)u = 0, \quad \text{on} \ \ \Omega \times (0,T)$$

where Ω is a bounded domain, $u = (u_1, u_2, u_3)$, and

$$(Eu)_i = \rho^{-1}\Big(\partial_i \lambda \nabla \cdot u + \sum_j \partial_j \mu (\partial_j u_i + \partial_i u_j)\Big),$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé parameters and $\rho > 0$ is the density.

We want to recover λ, μ and ρ from the DN map

$$\Lambda f = \Sigma_j \sigma_{ij}(u)\nu_j, \quad \text{on } \partial\Omega \times (0,T)$$

where ν is the outer normal and $\sigma_{ij}(u) = \lambda \nabla \cdot u \delta_{ij} + \mu(\partial_j u_i + \partial_i u_j)$ is the stress tensor.

The speed of P-waves is given by

$$c_p = \sqrt{(\lambda + 2\mu)/\rho}$$

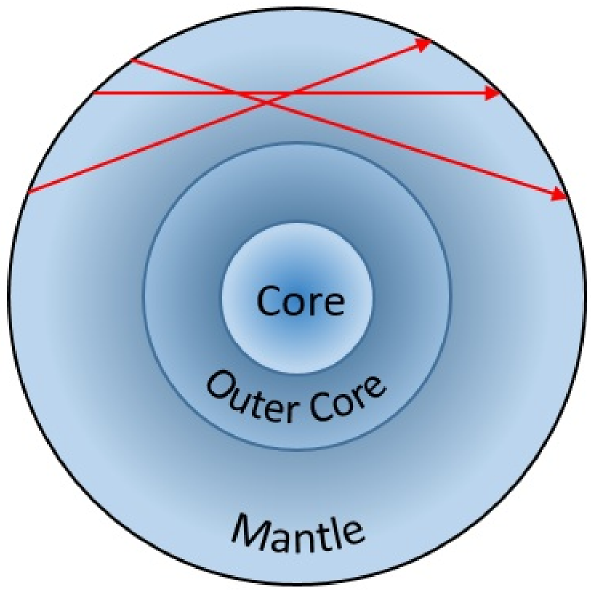
and the speed of S-waves is given by

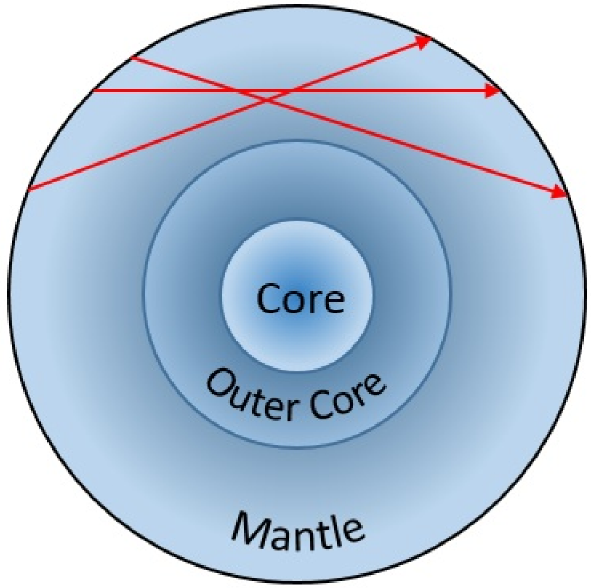
$$c_s = \sqrt{\mu/\rho}.$$

Rachelle has shown that one can recover the boundary jets and the coefficients inside if both speeds are simple. The proof of the later uses the boundary rigidity results for $c_p^{-2}dx^2$ and $c_s^{-2}dx^2$ and the inversion of the geodesic ray transform.

Unique continuation holds but the boundary control method does not work. The local problem was open.

Theorem (Stefanov-U-Vasy, 2017). If c_s and/or c_p increase with depth in $R_0 < |x| < R$, then knowing the normal derivative of the solution $u(t, x)$ on the boundary for all boundary values of $u(t, x)$ (i.e., for all boundary sources) recovers c_s and/or c_p uniquely there.

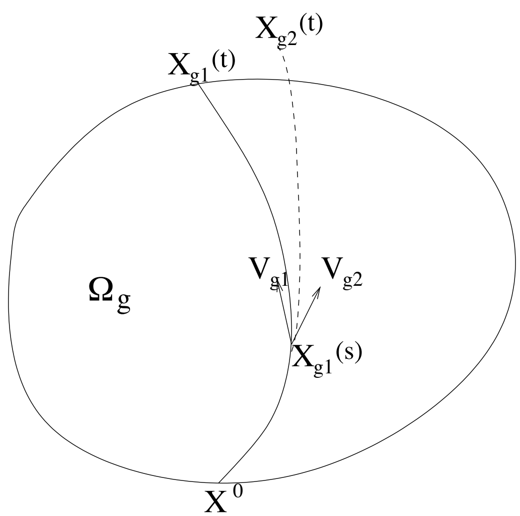




In particular, in the elastic Earth model, we can recover the pressure and the sheer speeds in the Mantle. The parameters jump across the interior boundary. It does not matter what happens inside (in the Outer Core, etc.); and there, the model may even change (liquid Outer Core). Under some conditions, we can determine also the location of discontinuities (Stefanov-U-Vasy, 2019, Caday-de Hoop-Katsnelson-U, 2019)

Second Step: Reduction to Pseudolinear Problem

Identity (Stefanov-U, 1998)



$$g_i = \frac{1}{c_i^2} dx^2,$$

$$T = d_{c_1},$$

$$F(s) = X_{c_2}\big(T - s, X_{c_1}(s, X^0)\big),$$

$$F(0) = X_{c_2}(T, X^0), \quad F(T) = X_{c_1}(T, X^0),$$

$$\int_0^T F'(s)ds = X_{c_1}(T, X^0) - X_{c_2}(T, X^0)$$

$$\int_0^T \frac{\partial X_{c_2}}{\partial X^0}\left(T - s, X_{c_1}(s, X^0)\right)(V_{c_1} - V_{c_2})\Big|_{X_{c_1}(s, X^0)} dS = X_{c_1}(T, X^0) - X_{c_2}(T, X^0)$$

Identity (Stefanov-U, 1998)

$$\int_0^T \frac{\partial X_{c_2}}{\partial X^0}\left(T-s, X_{c_1}(s,X^0)\right)(V_{c_1}-V_{c_2})\Big|_{X_{c_1}(s,X^0)} dS = X_{c_1}(T,X^0) - X_{c_2}(T,X^0)$$

$$V_{c_j} := \left(\frac{\partial H_{c_j}}{\partial \xi}, -\frac{\partial H_{c_j}}{\partial x}\right)$$ the Hamiltonian vector field.

$$(g_k) = \frac{1}{c_k^2}\left(\delta_{ij}\right), \quad k = 1,2$$

$$V_{g_k} = \left(c_k^2 \xi, \; -\frac{1}{2}\nabla(c_k^2)|\xi|^2\right)$$

Linear in c_k^2!

Reconstruction

$$\int_0^T \frac{\partial X_{c_1}}{\partial X^0}\Big(T-s, X_{c_2}(s,X^0)\Big)\times \Big((c_1^2-c_2^2)\xi, -\frac{1}{2}\nabla(c_1^2-c_2^2)|\xi|^2\Big)\Big|_{X_{c_2}(s,X^0)} dS = \underbrace{X_{c_1}(T,X^0)}_{\text{data}} - X_{c_2}(T,X^0)$$

Inversion of weighted geodesic ray transform and use similar methods to U-Vasy.