Three-dimensional Motion Reconstruction from Parallel-Beam Projection Data

Denise Schmutz

Modern Challenges in Imaging
In the Footsteps of Allan Cormack
Tufts University

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Joint work with Peter Elbau, Monika Ritsch-Marte and Otmar Scherzer.

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FWF Special Research Program F68
Tomography Across the Scales
tomography.csc.univie.ac.at

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Optical microscopy of trapped objects

Trapping is a tool for holding and moving microscopic particles in a contact-free and non-invasive manner.

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Thalhammer, Steiger, Meinschad, Hill, Bernet, and Ritsch-Marte “Combined acoustic and optical trapping” 2011

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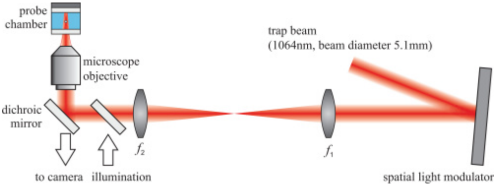

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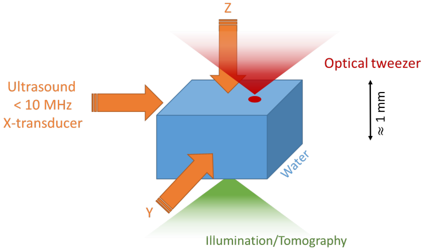

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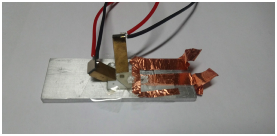

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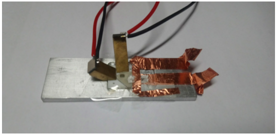

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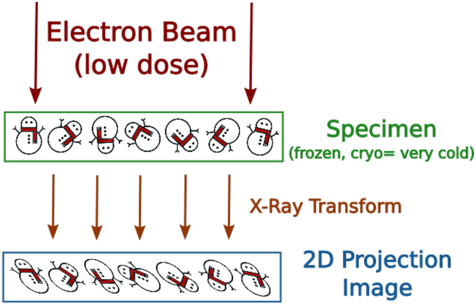
Relation to single particle cryo-electron microscopy



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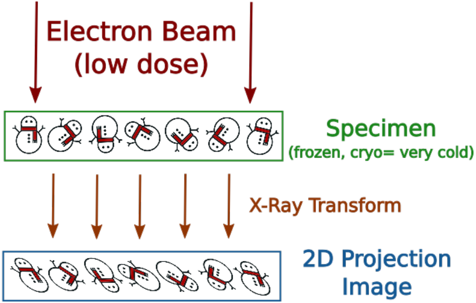

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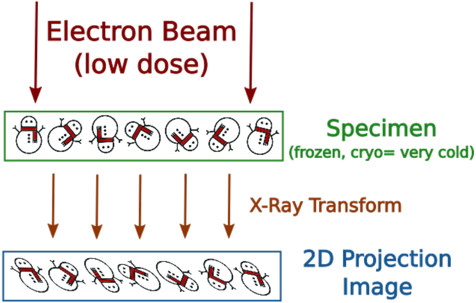

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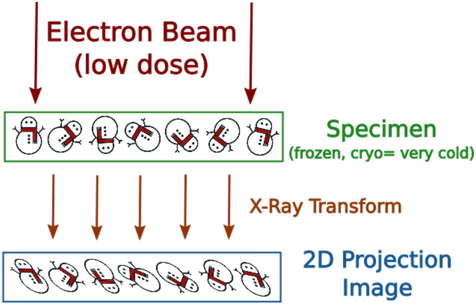

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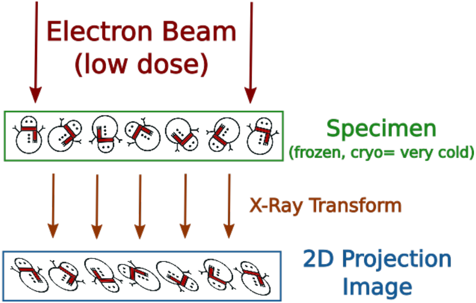

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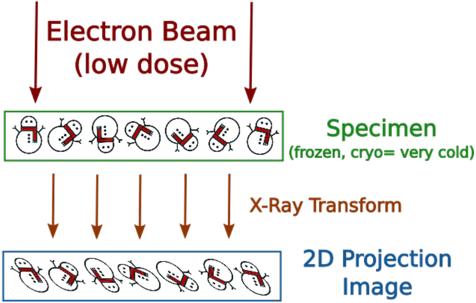

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- Orientation reconstruction via *common line method*

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- Bounded object in $\mathbb{R}^3$

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$$A(t,x) = \mathcal{C}_3 + R(t)(x - \mathcal{C}_3 + T(t))$$

$R \in C(\mathbb{R}; SO(3)) \ldots$ rotation
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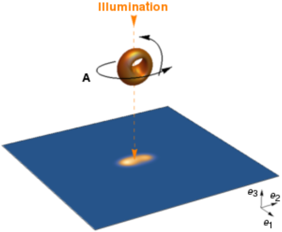
- Object is illuminated from the e_3-direction with a uniform intensity
- Light moves along straight lines and only suffers from attenuation

Measurements



Attenuation projection mappings $\mathcal{J}$

$$(T, R) \mapsto \mathcal{J}[T, R](t, x_1, x_2) = \int_{-\infty}^{\infty} u\big(A(t, x)\big)\, \mathrm{d}x_3$$

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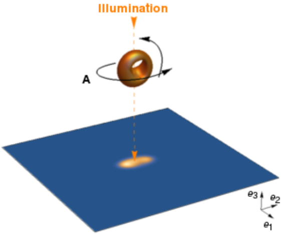

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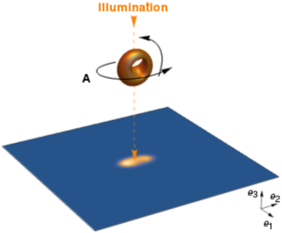

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Goal

Reconstruction of $R(t)$ and $T(t)$ from collected data of $\mathcal{J}[T, R](t, x_1, x_2)$.

Elbau, Ritsch-Marte, Scherzer, and Schmutz *Inverse Problems of Trapped Objects* 2019

Formulation in Fourier space

- n-dimensional Fourier transform

$$\mathcal{F}_n[f](k) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} f(x) \mathrm{e}^{-\mathrm{i}\langle k,x\rangle} \, \mathrm{d}x$$

- Orthogonal projection $P : \mathbb{R}^3 \to \mathbb{R}^2, Px = \binom{x_1}{x_2}$
- Its adjoint $P^T : \mathbb{R}^2 \to \mathbb{R}^3, P^T k = \binom{k}{0}$

Natterer *The mathematics of computerized tomography* 2001

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Lemma 1

Let $u \in C_c(\mathbb{R}^3;\mathbb{R})$ and $\mathcal{J}[R,T]$ be the attenuation mapping of a rigid body motion (R,T). Then, the following identity holds:

$$\mathcal{F}_2[\mathcal{J}[T,R]] = \sqrt{2\pi}\, \mathcal{F}_3[u](R(t)P^T k)\, \mathrm{e}^{\mathrm{i}\left\langle R(t)P^T k, \mathcal{C}_3\right\rangle} \mathrm{e}^{\mathrm{i}\langle k, P(T(t)-\mathcal{C}_3)\rangle}.$$

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- Similar to the *projection-slice theorem*

Natterer *The mathematics of computerized tomography* 2001

1. Motivation

2. Mathematical Model

3. Motion Estimation
   - Reconstruction of the translation
   - Reduced attenuation maps
   - Reconstruction of the rotation

4. Numerics

Reconstruction of the translation T

- It is not possible to reconstruct the translation along the e_3-direction. For $\rho \in C(\mathbb{R};\mathbb{R})$ it holds that

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$$\mathcal{C}_2(t) := \frac{1}{\int_{\mathbb{R}^2} \mathcal{J}[T,R](t,x)\,\mathrm{d}x} \int\limits_{\mathbb{R}^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mathcal{J}[T,R](t,x)\,\mathrm{d}x.$$

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Let $u \in C_c(\mathbb{R}^3;\mathbb{R})$ and $\mathcal{J}$ be the attenuation mapping of a rigid motion of u. Then,

$$P(\mathcal{C}_3 - T(t)) = \mathcal{C}_2(t)$$

for every $T \in C(\mathbb{R};\mathbb{R}^3), R \in C(\mathbb{R};SO(3)), t \in \mathbb{R}$.

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- If we start the motion at time $t = 0$ with the normalisaton $T(0) = 0$, we have

$$P(T(t)) = \mathcal{C}_2(0) - \mathcal{C}_2(t)$$

Reduced attenuation map

- From Lemma 1

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- Easy to get rid of the dependence on T
- We define the *reduced attenuation map* corresponding to u as

$$\begin{aligned}\tilde{\mathcal{J}} : \mathbb{R}\times\mathbb{R}^2 &\to \mathbb{R}\\ (t,k) &\mapsto \mathcal{F}_2[\mathcal{J}[T,R]](t,k)\,\mathrm{e}^{\mathrm{i}\langle k,\mathcal{C}_2\rangle}\end{aligned}$$

- $\tilde{\mathcal{J}}$ only depends on R

Symmetry property of reduced attenuation map

Lemma 2

Let $u \in C_c(\mathbb{R}^3;\mathbb{R})$ *and let* $\tilde{\mathcal{J}}$ *be the corresponding reduced attenuation map. Then, for arbitrary* $R \in C(\mathbb{R}; SO(3))$ *the following identity holds*

$$\tilde{\mathcal{J}}\left(s, \frac{\lambda}{t-s} P(e_3 \times (R(s)^T R(t) e_3))\right) = \tilde{\mathcal{J}}\left(t, \frac{\lambda}{s-t} P(e_3 \times (R(t)^T R(s) e_3))\right)$$

for all $\lambda \in \mathbb{R}$ *and* $s, t \in \mathbb{R}$ *with* $s \neq t$.

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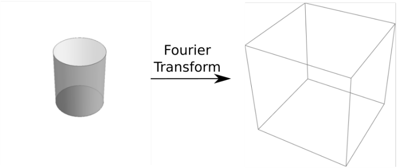

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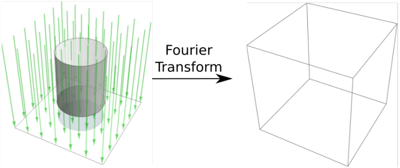

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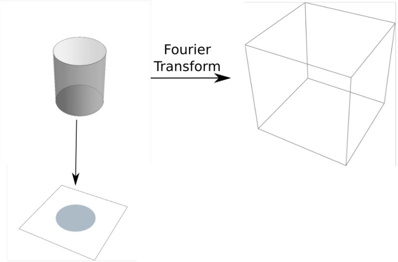

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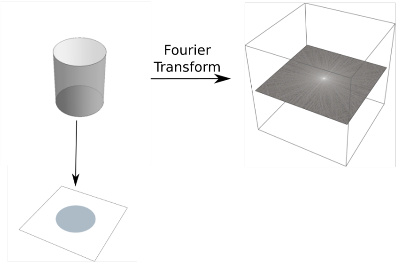

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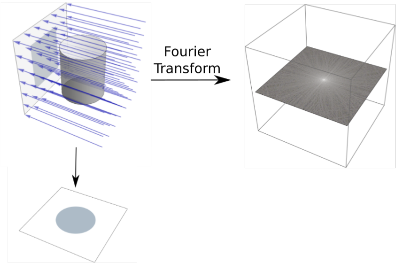

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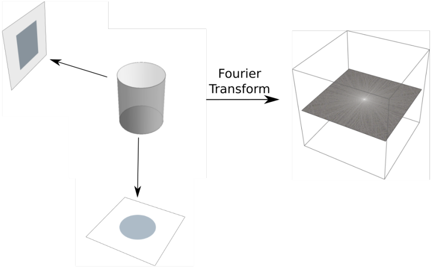

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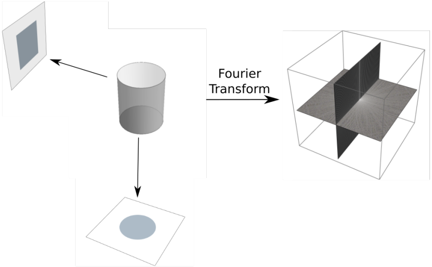

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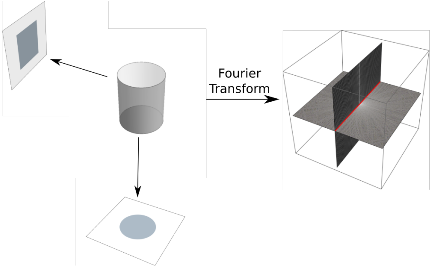

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  - For $i = 1$ and $g : \mathbb{R} \to \mathbb{R}^2, t \mapsto (g_1(t), g_2(t))^T$ this is for example
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How to reconstruct

- Consider function

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- The value of this constant function will then be $\omega_3(s)$

Reconstruction of the cylindrical radius α

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where

$$\begin{aligned} A_0(\mu) = \frac{1}{4}\mu(\omega_3+\sigma)\Big[&\mu^2\omega_3(\omega_3-\sigma)\mathbf{D}^3[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp, v^\perp, v^\perp]\!] \\ &+ 2\mu\mathbf{D}^2[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp, \omega_3\sigma v - \omega_3' v^\perp]\!] + 2\mathbf{D}^1[\tilde{\mathcal{J}}](s,\mu v)[\![\omega_3^2 v^\perp + \omega_3' v]\!] \\ &- \mu(3\omega_3-\sigma)\partial_t\mathbf{D}^2[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp, v^\perp]\!] + 2\partial_{tt}\mathbf{D}^1[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp]\!]\Big], \\ A_{02}(\mu) = \frac{1}{2}\mu(\omega_3+\sigma)&\mathbf{D}^1[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp]\!], \\ A_1(\mu) = \frac{1}{2}(\omega_3+\sigma)\Big[&\mu\omega_3\mathbf{D}^2[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp, v^\perp]\!] - \omega_3\mathbf{D}^1[\tilde{\mathcal{J}}](s,\mu v)[\![v]\!] - \partial_t\mathbf{D}^1[\tilde{\mathcal{J}}](s,\mu v)[\![v^\perp]\!]\Big]. \end{aligned}$$

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- Consider $(\star)$ as an overdetermined linear system for $\alpha^2(t)$ and $\frac{\alpha'(t)}{\alpha(t)}$

Uniqueness considerations

- Similar non-uniqueness issue as in Cryo-EM

Lamberg "Unique recovery of unknown projection orientations in three-dimensional tomography" 2008

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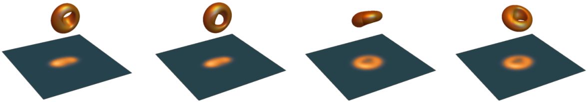

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Simulations

- For the points $P_1 = (1, \frac{1}{2}, -1)$, $P_2 = (-\frac{1}{2}, 1, 1)$, $P_3 = (0, -1, \frac{1}{2})$ and the diagonal matrix $D = \operatorname{diag}(\sqrt{2}, 1, 1)$ we consider as an example the attenuation coefficient

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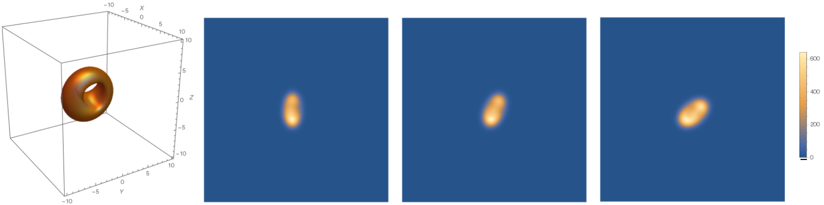

- Motion

$$T(t) = \begin{pmatrix} \cos(6t)\cos(12t) \\ \cos(6t)\sin(12t) \\ \sin(t) \end{pmatrix} \text{ and } \omega(t) = \begin{pmatrix} \alpha(t)\cos(\varphi(t)) \\ \alpha(t)\sin(\varphi(t)) \\ \omega_3(t) \end{pmatrix}$$

with

$$\alpha(t) = 1 + 10t^2, \;\; \varphi(t) = \pi t + \frac{\pi}{3}, \text{ and } \omega_3(t) = \frac{1}{2} + \sqrt{\frac{1}{2} + 5t}.$$

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- Discretisation in space

$$(j_1, j_2, j_3)\delta_x,\ j \in \{-512, \dots, 511\}^2 \times \{-256, \dots, 255\} \text{ with } \delta_x = 0.05$$

and in time

$$\ell\,\delta_t,\ \ell \in \{0, \dots, 999\} \text{ for } \delta_t = 0.0005.$$

Reconstruction procedure I

- Calculate the center of the attenuation projection images and read off the first two components of the **displacement** $T(t)$ via

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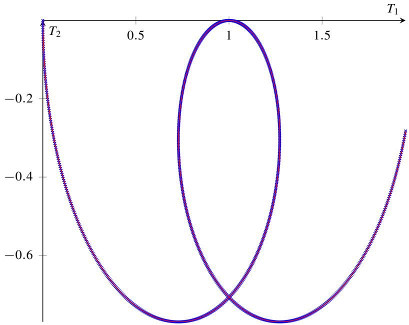

Reconstruction procedure II

- Consider least square minimisation problem for the function

$$\tilde{\omega}_3(\varphi(t)) = \underset{\omega_3}{\operatorname{argmin}} \left\{ \sum_{j=-512}^{511} \left| \partial_t \tilde{J}\left(t, j\delta_x \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix}\right) - \omega_3(t) \mathbf{D}^1[\tilde{J}]\left(t, j\delta_x \begin{pmatrix} \cos(\varphi(t)) \\ \sin(\varphi(t)) \end{pmatrix}\right) \left[j\delta_x \begin{pmatrix} -\sin(\varphi(t)) \\ \cos(\varphi(t)) \end{pmatrix} \right] \right|^2 \right\}$$

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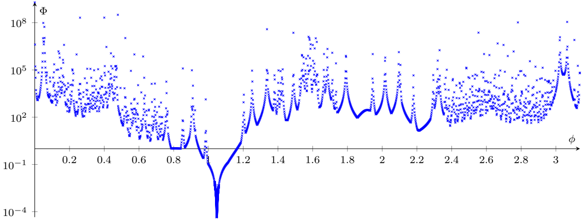

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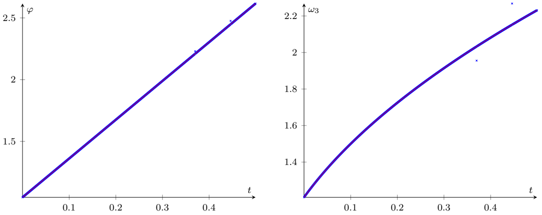

Reconstruction procedure III

- To obtain **the cylindrical radius** α, we consider

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as overdetermined linear system (one equation for each value $\mu \in \{j\delta_x \mid j \in \{-512, \ldots, 511\}\}$) for $\alpha^2(t)$ and $\frac{\alpha'(t)}{\alpha(t)}$, where the coefficients can be explicitly calculated with the values of φ and ω_3 obtained so far.

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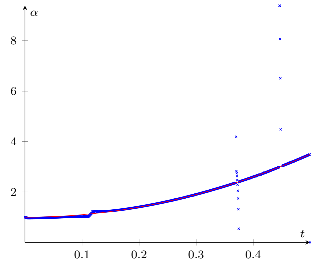
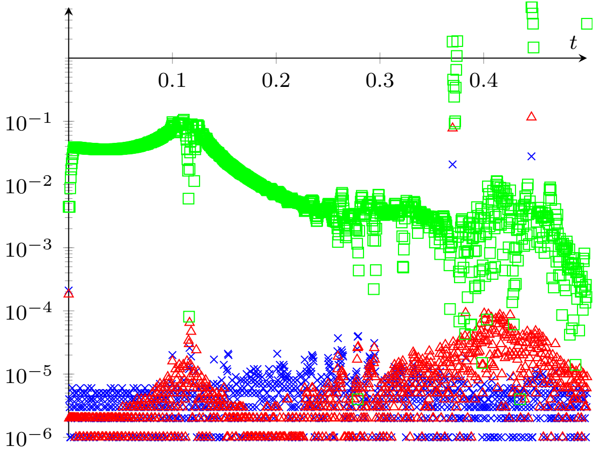


Errors

Absolute errors in the reconstructions of φ (the crosses), ω_3 (the triangles) and α (the squares).



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- Study corrections or alternative approaches required when going from attenuation projection images to optical images

References

- G. Thalhammer, R. Steiger, M. Meinschad, M. Hill, S. Bernet, and M. Ritsch-Marte Combined acoustic and optical trapping *Biomed. Opt. Express* 2.10 (2011).
- M. van Heel Angular reconstitution: A posteriori assignment of projection directions for 3D reconstruction *Ultramicroscopy* 21.2 (1987).
- A. B. Goncharov Methods of integral geometry and recovering a function with compact support from its projections in unknown directions *Acta Appl. Math.* 11.3 (1988).
- P. Elbau, M. Ritsch-Marte, O. Scherzer, and D. Schmutz *Inverse Problems of Trapped Objects* arXiv:1907.01387 (2019).
- F. Natterer The mathematics of computerized tomography. Classics in Applied Mathematics vol. 32 (2001).
- L. Lamberg Unique recovery of unknown projection orientations in three-dimensional tomography *Inverse Probl. Imaging* 2.4 (2008).

Thank you for your attention!

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