Generalized Radon transforms with cusp singularities

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Overview

- Generalized Radon transforms over curves $\gamma(t) = (t, t^n, t^m)$
- Cases $n = 2, m = 3, m = 4$
- FIOs with singularities
- Composition calculus

Generalized Radon transforms

- X, Y dim n, $\quad Z \subset X \times Y$ dim $n+k$

$$\begin{array}{ccc} & Z & \\ \pi_X \swarrow & & \searrow \pi_Y \\ X & & Y \end{array}$$

- $Y_x = \pi_y \pi_x^{-1}(\{x\}) \subset Y$
- $Rf(x) = \int_{Y_x} f(y)dy$

- $Y_x = \{\gamma(x,t), t \in R\}$ are curves

- $Z = \{(x, \gamma)\}$

- X-ray transform

Generalized Radon transforms

- Greenleaf, Seeger, Wainger
- convolution with measure $\mu = \psi(t)dt$ supported on curve $(t, t^2, \ldots, t^n)$.
- $Rf(x) = \int_{\mathbf{R}} f(x - (t, t^2, \ldots, t^n))\psi(t)dt = \int e^{i[(x_2 - y_2 + (x_1 - y_1)^2)\theta_2 + \cdots + (x_n - y_n + (x_1 - y_1)^n)\theta_n]} f(y)a(x, y, \theta)d\theta dy$
- $Z = \{(x, y) : x_i - y_i + (x_1 - y_1)^i = 0\}$

- $\gamma(t) = (t, t^n, t^m),\ n < m$
- $Rf(x) = \int_{\mathbf{R}} f(x - (t, t^n, t^m))\psi(t)dt = \int e^{i[(x_2 - y_2 + (x_1 - y_1)^n)\theta_1 + (x_3 - y_3 + (x_1 - y_1)^m)\theta_2]} f(y)a(x, y, \theta)d\theta dy$
- FIOs with singularities depending on n, m

Applications

- Monostatic SAR: plane-trajectory-antenna
- $\gamma(s) = (s, s^3, h)$
- Single source seismology: accoustic waves-pressure field
- fold/cusp caustics
- F : image $\rightarrow$ data is known
- F is FIO with singularities
- to find the image F^*F

Fourier Integral Operators

- $F : \mathcal{E}'(Y) \to \mathcal{D}'(X)$

$$Ff(x) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta) f(y) d\theta dy$$

- ϕ is a nondegenerate **phase function**
- a is a **symbol** S^M: $|\partial^\alpha_{x,y} \partial^\beta_\theta a| < c(1+|\theta|)^{M-|\beta|}$
- C is a **canonical relation** in $T^*X \setminus 0 \times T^*Y \setminus 0$

$$C = \{(x, d_x\phi; y, -d_y\phi); d_\theta\phi = 0\}$$

- $I^m(C)$, $m = M + \frac{N}{2} - \frac{n_X + n_Y}{4}$
- Adjoint $F^* f(y) = \int e^{-i\phi(x,y,\theta)} \bar{a}(x,y,\theta) f(x) d\theta dx$
- If $F \in I^m(X, Y, C)$ then $F^* \in I^m(Y, X, C^t)$

Examples

- $Q : \mathcal{E}'(X) \to \mathcal{D}'(X)$ Pseudodifferential operator
- $\phi(x, y, \theta) = (x - y) \cdot \theta$
- $C = \{(x, d_x\phi; y, -d_y\phi); d_\theta\phi = 0\} = \{(x, \theta; y, \theta)| \ \ x = y\} =$ $\Delta =$diagonal in $T^*X \times T^*X$

- $Q : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ FIO associated to a canonical graph
- $\phi(x, y, \theta) = \psi(x, \theta) - y \cdot \theta$
- $C = \{(x, d_x\psi; y, \theta); d_\theta\psi = y\} = Gr(\chi)$

Analysis

- Geometry of $C \in T^*X \setminus 0 \times T^*Y \setminus 0$

$$
\begin{array}{ccc}
 & \pi_L \swarrow \quad \searrow \pi_R & \\
T^*X \setminus 0 & & T^*Y \setminus 0
\end{array}
$$

- π_L, π_R are local diffeomeorphisms: F^*F a FIO
- $\Sigma = \{(x, y, \theta) \in C$; det d$\pi_L$ =det d$\pi_R = 0\}$
- singularities: folds; cusps; blowdowns; one sided; two sided

Fold/Cusp singularities

- **Whitney Folds**

  $f : R^n \to R^n$ has a fold singularity along $\Sigma = \{x : det\ df = 0\}$ if Σ is smooth and if Ker $df \not\subseteq T\Sigma$.

- Ex: $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_n^2)$
- $\Sigma = \{x_n = 0\}$; Ker $df = \frac{\partial}{\partial x_n}$

- **Cusps**

  $f : R^n \to R^n$ has a cusp singularity along $\Sigma = \{x : det\ df = 0\}$ if Σ is smooth, if Ker $df \subset T\Sigma$ along Σ_1.

- Ex: $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-1}x_n + x_n^3)$
- $\Sigma = \{x_{n-1} + 3x_n^2 = 0\}, \quad \Sigma_1 = \{x_{n-1} + 3x_n^2 = 0 = x_n\}$
- Ker $df = \frac{\partial}{\partial x_n}$

Singularities of the generalized Radon transforms

- $\phi(x, y, \theta_2, \theta_3) = (x_2 - y_2 + (x_1 - y_1)^n)\theta_2 + (x_3 - y_3 + (x_1 - y_1)^m)\theta_3$

- $C = \{(x_1, x_2, x_3, n(x_1 - y_1)^{n-1}\theta_2 + m(x_1 - y_1)^{m-1}\theta_3, \theta_2, \theta_3;$
$y_1, y_2, y_3, n(x_1 - y_1)^{n-1}\theta_2 + m(x_1 - y_1)^{m-1}\theta_3, \theta_2, \theta_3);$
$x_2 - y_2 + (x_1 - y_1)^n = 0; x_3 - y_3 + (x_1 - y_1)^m = 0\}$

- $\Sigma = \{(x_1 - y_1)^{n-2}(n(n-1) + m(m-1)(x_1 - y_1)^{m-n}) = 0\}$

- Σ smooth for $n = 2$

- Ker $\pi_L = \frac{\partial}{\partial y_1}$, Ker $\pi_R = \frac{\partial}{\partial x_1}$

- $m = 3, \gamma(t) = (t, t^2, t^3)$, both π_L, π_R have fold singularities

- $m = 4, \gamma(t) = (t, t^2, t^4)$, *both* π_L, π_R *have cusp singularities* (RF, Greenleaf)

- $m \geq 5$, no stable class of singularities

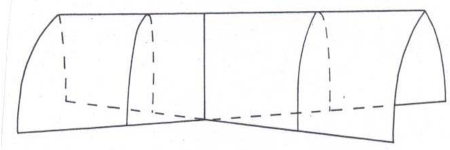
Singularities in Inverse problems

- Singularities in monostatic SAR
- γ has nonzero curvature: both π_L, π_R have fold singularities (RF, Nolan, Cheney)
- γ has zero curvature: π_L fold; π_R blowdown (RF, Nolan, Cheney)
- *curvature of γ has simple zeros: π_L fold; π_R cusp (RF, Nolan)*

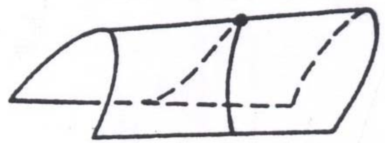
- Singularities in seismology
- fold caustics: both π_L, π_R have fold singularities (RF, Nolan)
- *cusp caustics: both π_L, π_R have cusp singularities (RF, Greenleaf)*

Open umbrella

- Ex $g : R^2 \to R^3,\ g(x,y) = (x^2, y, xy)$ (cross-cap)(Guillemin)



- Ex $U : R^2 \to R^4,\ U(x,y) = (x^2, y, xy, \frac{2}{3}x^3)$ (Givental)



- Ker $dU = \partial_x$; $\Sigma = \{x = y = 0\}$

F^*F cusp/cusp

- Generalized Radon transform over $\gamma(t) = (t, t^2, t^4)$
  $\chi(x, y, \theta_2, \theta_3) = (x_2 - y_2 - (x_1 - y_1)^2)\theta_2 + (x_3 - y_3 - (x_1 - y_1)^4)\theta_3$
- $\pi_L,\ \pi_R$ have cusp singularities
- (1) same cusp points
- (2) images of the cusp points are involutive
- $F^*F \to C^t \circ C = \Delta \cup \tilde{C},\ \tilde{C} =$ **open umbrella**
- **The image of a map $\psi : R^n \to R^{2n}$, drops rank simply at Σ codimension 2 and Ker $d\psi \not\subseteq T\Sigma$, lagrangian away from Σ**
- (RF, A. Greenleaf) Let $C \subset T^*X \times T^*Y$ be a two-sided cusp. If $F \in I^m(C)$ then $F^*F \in I^{2m}(\Delta, \tilde{C})$ where $\tilde{C}$ is an open umbrella.

F^*F Fold/cusp

- RF, Nolan
- $\phi_{model} = (x' - y')\theta' + (x_{n-1}x_n + x_n^3)y_n\theta_1 + x_n y_n^2 \theta_1$
- π_L has fold singularities; π_R has cusp singularities
- images of the cusp points are symplectic
- $F^*F \to C^t \circ C = \Delta \cup \tilde{C}$
- Let $\tilde{C} \subset T^*X \times T^*Y$ is a fold/cusp canonical relation, and $A, B : E'(Y) \to E'(X)$ are properly supported FIOs associated to $\tilde{C}$ of orders $m, m' \in R$, resp., then $B^*A \in I^{m+m'}(\Delta, \tilde{C})$ where $\tilde{C}$ is an open umbrella.

Weak normal form

- Normal form for a two sided fold: Melrose, Taylor

- Weak normal forms: Greenleaf, Uhlmann, RF, Marhuenda

- $\chi(x, y, \theta_2, \theta_3) = (x_2 - y_2 - (x_1 - y_1)^2)\theta_2 + (x_3 - y_3 - (x_1 - y_1)^4)\theta_3$

- Any two sided cusp canonical relation with properties 1-2

  $\chi(x, y, \theta_2, \theta_3) = (x_3 - y_3)\theta_3 + (x_1 - y_1)^4 S_3 + (S_2 - y_2 + (x_1 - y_1)^2 S_4)\theta_2$
  where $\partial_{x_2} S_2, S_3, S_4 \neq 0$

- $\phi_{model} = (x' - y')\theta' + (x_{n-1}x_n + x_n^3)y_n\theta_1 + x_n y_n^2 \theta_1$

- Any fold/cusp canonical relation can be parametrized by

- $\tilde{\phi} = (x' - y') \cdot \theta' + x_n y_n^2 \theta_1 + (x_n x_{n-1}\theta_1 + x_n^3 S(\cdot))y_n + N(\cdot)$

Other singularities

- **Morin singularities:** S_{1_k}
- $f(x_1, x_2, \ldots, x_n) = (x_1, x_2, \ldots, x_{n-k+1}x_n + \cdots + x_{n-1}x_n^{k-1} + x_n^{k+1})$
- $\Sigma = \{x_{n-k+1} + \cdots + (k-1)x_{n-1}x_n^{k-2} + (k+1)x_n^k = 0\}$
- Ker $df = \frac{\partial}{\partial x_n}$
- If X, Y are n-dimensional manifolds, $\tilde{C} \subset T^*X \times T^*Y$ is a canonical relation with π_R a cusp and π_L with S_{1_k} singularity, and $A, B : E'(Y) \to E'(X)$ are properly supported FIOs associated to $\tilde{C}$ of orders $m, m' \in R$, resp., then $WF(B^*A) \subset \Delta \cup \tilde{\Lambda}$.

H^s Estimates

- (Greenleaf, Seeger) one cusp: $F \in I^m(C)$ then $F : H^s \to H^{s-m-\frac{1}{3}}$
- (Comech) fold/cusp: $F \in I^m(C)$ then $F : H^s \to H^{s-m-\frac{1}{5}}$
- (Melrose) two sided fold: $F \in I^m(C)$ then $F : H^s \to H^{s-m-\frac{1}{6}}$
- (Greenleaf, Uhlmann) one sided fold: $F \in I^m(C)$ then $F : H^s \to H^{s-m-\frac{1}{4}}$