Radon transforms supported in hypersurfaces

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Plan of talk

The interior Radon transform

Distributions f such that the Radon transform Rf is supported in a hypersurface

Theorem. If there exists a compactly supported distribution f such that Rf is supported in the set of tangents to the boundary of a domain D, then D must be an ellipse.

A conjecture of Arnold

Sketch of proof of the theorem

The plane Radon Transform

The 2-dimensional Radon transform integrates a compactly supported function f over lines L

$$Rf(L) = \int_L f\,ds.$$

Here L is a line in the plane and ds is length measure on L. Occasionally I shall use the familiar parametrisation

$$Rf(\omega, p) = \int_{x\cdot\omega=p} f\,ds, \quad (\omega, p) \in S^1 \times \mathbb{R},$$

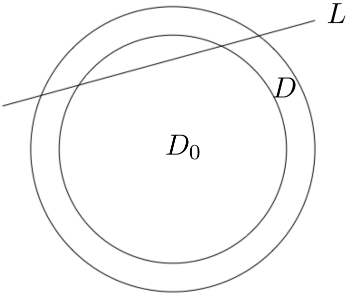
where the line L is defined by $x \cdot \omega = p$. Clearly

$$Rf(\omega, p) = Rf(-\omega, -p).$$

The Interior Radon Transform

Given two concentric disks D and $\overline{D_0} \subset D$ it is well known that there exists a non-trivial function f with support in $\overline{D}$ such that

$$Rf(L) = 0 \quad \text{for all lines } L \text{ that meet } D_0.$$

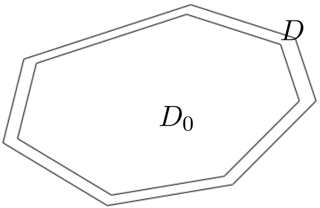


In fact one can take f radial, that is, $f(x) = f(r)$ with $r = |x|$. One can prescribe $g(p)$ arbitrarily and find $f(r)$ so that $Rf(p) = g(p)$, for instance choose $g(p) = 0$ for $|p| \leq p_0 < 1$.

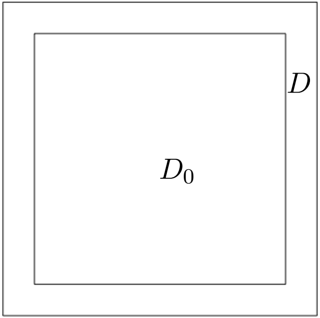
The Interior Radon Transform, cont.

It is natural to replace the disks by arbitrary convex sets.

Conjecture. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f, not identically zero, $\operatorname{supp} f = \overline{D}$, such that its Radon transform $Rf(L)$ vanishes for every line L that intersects D_0.



Example:



A Radon transform supported on a curve

Let f_0 be the function in the plane defined by

$$f_0(x) = \frac{1}{\pi} \frac{1}{\sqrt{1 - |x|^2}} \quad \text{for } |x| < 1$$

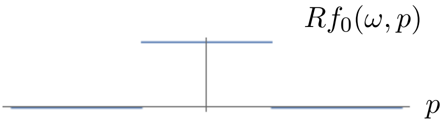
and $f = 0$ for all other $x = (x_1, x_2)$. An easy calculation shows that

$$Rf_0(\omega, p) = \int_{x \cdot \omega = p} f_0(x)\, ds = 1 \quad \text{for } |p| < 1,$$

and obviously $Rf_0(\omega, p) = 0$ for $|p| \geq 1$.

Let f be the distribution $f = \Delta f_0 = (\partial_{x_1}^2 + \partial_{x_2}^2) f_0$.

Now use the well known formula $R(\Delta h)(\omega, p) = \partial_p^2 Rh(\omega, p)$ with $h = f_0$.

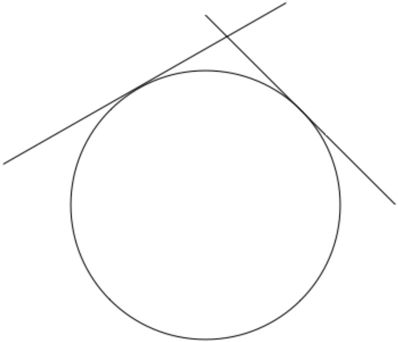


It follows that

$$Rf(\omega, p) = \delta'(p+1) - \delta'(p-1),$$

if $\delta(p)$ denotes the Dirac measure at the origin.

This means that the distribution $f = \Delta f_0$ has the property that its Radon transform, a distribution on the manifold of lines in the plane, must be supported on the set of tangents to the unit circle.



The Radon transform of a *distribution* f in $\mathbb{R}^n$ is defined by

$$\langle Rf, \varphi \rangle = \langle f, R^*\varphi \rangle, \quad \text{for all test functions } \varphi\text{, where}$$

$$(R^*\varphi)(x) = \int_{S^{n-1}} \varphi(\omega, x \cdot \omega) d\omega,$$

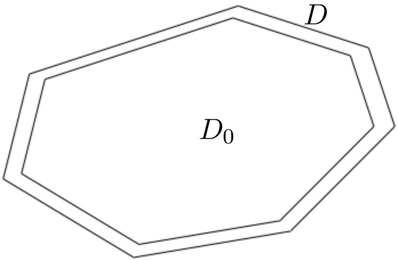
$d\omega$ is surface measure on S^{n-1}, or

$$(R^*\varphi)(x) = \int_{L \ni x} \varphi(L) d\mu(L).$$

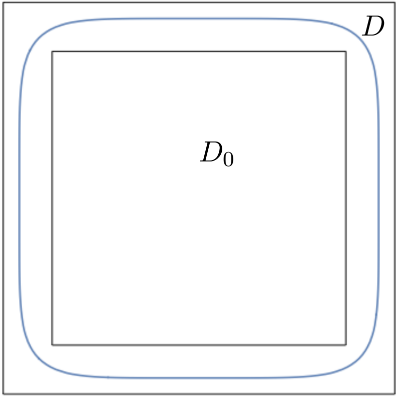
By means of an affine transformation we can easily construct a similar example where D is an ellipse.

Now back to our Conjecture:

Conjecture. Let D and D_0 be bounded convex domains in the plane with $\overline{D_0} \subset D$. Then there exists a smooth function f, not identically zero, supported in D, such that its Radon transform $Rf(L)$ vanishes for every line L that intersects D_0.



Proof idea for Conjecture: find a compactly supported distribution f whose Radon transform is supported on the set of tangents to the blue curve.



However: to my surprise I found the following:

Theorem 1 (JB 2018). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

If ∂D is C^1 smooth, the supporting planes for D are of course tangent planes to ∂D.

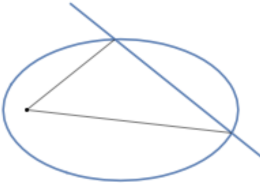
Newton's lemma

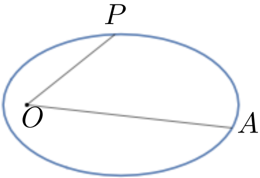
A bounded domain in the plane is called *algebraically integrable*, if the area of a segment cut off by a secant line is an algebraic function of the parameters defining the line.

Lemma 28 in Newton's *Principia* reads according to Arnold and Vassiliev in *Newton's Principia read 300 years later* (Notices of the AMS 1989):

Lemma. There exists no algebraically integrable convex non-singular algebraic curve.

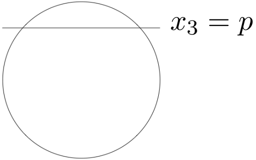
Newton's Lemma, cont..





A segment is equal to a sector minus a triangle, and the area of the triangle depends algebraically on the coordinates of the corners.

Higher dimensions: the case of odd dimension



The volume of the part of the unit ball in $\mathbb{R}^3$ that lies above the plane $x_3 = p$ is

$$\int_p^1 \pi(\sqrt{1-t^2})^2 dt = \int_p^1 \pi(1-t^2)dt = \frac{\pi}{3}(p^3 - 3p + 2).$$

So the volume function $V(p)$ is not only algebraic but polynomial.

Same for arbitrary odd dimension.

And same for ellipsoids.

Arnold’s Conjecture

Problem 1987-14 in *Arnold’s Problems* reads:

Do there exist smooth hypersurfaces in $\mathbb{R}^n$ (other than the quadrics in odd-dimensional spaces), for which the volume of the segment cut by any hyperplane from the body bounded by them is an algebraic function of the hyperplane?

The case of even dimension

Theorem 2. (Vassiliev 1988) There exist no convex algebraically integrable bounded domains in even dimensions.

V. A. Vassiliev: *Applied Picard - Lefschetz Theory*, AMS 2002.

A summary

n even: $p \mapsto V(\omega, p)$ is never algebraic (Vassiliev)

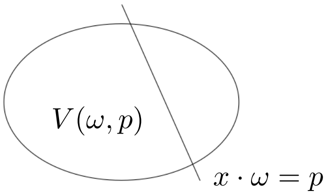
n odd, ∂D ellipsoid: $p \mapsto V(\omega, p)$ is polynomial

n odd, ∂D not ellipsoid: unknown if $p \mapsto V(\omega, p)$ can be algebraic

The case of odd dimension

Since Arnold's conjecture is still unsolved in this case, one has considered a weaker statement, namely:

Denote by $V(\omega, p)$ the volume cut out from the domain D by the hyperplane $x \cdot \omega = p$. Assume that $p \mapsto V(\omega, p)$ is a *polynomial* for every ω. Prove that the boundary of D must be an ellipsoid.



Theorem 3. (Koldobsky, Merkurjev, and Yaskin 2017) Assume that D is convex and has C^∞ boundary and that $p \mapsto V(\omega, p)$ is a polynomial of degree $\leq N$ for every ω. Then the boundary of D must be an ellipsoid.

Recall:

Theorem 1 (JB 2018). Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. Assume that there exists a distribution $f \neq 0$, supported in $\overline{D}$, such that Rf is supported in the set of supporting planes to ∂D. Then the boundary of D is an ellipsoid.

Theorem 1 implies Theorem 3

Let $\chi_D(x)$ be the characteristic function for the domain D and let $V(\omega, p)$ be the volume function discussed earlier.

It is clear that

$$\partial_p V(\omega, p) = \partial_p \int_{x\cdot\omega<p} \chi_D(x)dx = (R\chi_D)(\omega, p).$$

Applying the formula $R(\Delta h)(\omega, p) = \partial_p^2 Rh(\omega, p)$ to $h = \chi_D$ and iterating gives for every k

$$R(\Delta^k \chi_D)(\omega, p) = \partial_p^{2k} R\chi_D(\omega, p).$$

If $p \mapsto V(\omega, p)$ is polynomial (for p such that the plane $x \cdot \omega = p$ intersects D) then $p \mapsto R(\chi_D)(\omega, p)$ is polynomial, so $\partial_p^{2k} R\chi_D(\omega, p) = 0$ if k is large enough except at the jump points, which correspond to tangent planes. So

$$f = \Delta^k \chi_D$$

has the property that its Radon transform is supported on the set of tangent planes to ∂D. By Theorem 1 it follows that ∂D is an ellipsoid.

Remark 1. Theorem 1 implies Theorem 3 without the smoothness assumption on the boundary of D.

Remark 2. Theorem 3 shows that the Radon transform of the characteristic function χ_D cannot be polynomial unless ∂D is an ellipsoid. Theorem 1 shows that *no function* supported in D can have a polynomial Radon transform unless ∂D is an ellipsoid.

Distributions supported on the set of supporting planes

Assume for simplicity that $D = -D$. Let $\rho(\omega)$ be the supporting function for D

$$\rho(\omega) = \sup\{x \cdot \omega;\, x \in D\}.$$

The hyperplane $x \cdot \omega = p$ is a supporting plane to ∂D if and only if

$$p = \rho(\omega) \qquad \text{or} \qquad p = -\rho(\omega).$$

If $q(\omega)$ is an even function on S^{n-1}, then

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big)$$

satisfies $g(\omega, p) = g(-\omega, -p)$, and hence defines a distribution (of order zero) on the manifold of hyperplanes. More generally, if $g = Rf$, f compactly supported, and g is supported on $p = \pm\rho(\omega)$, then $g(\omega, p)$ can be written

$$g(\omega, p) = \sum_{j=0}^{m-1} q_j(\omega)\big(\delta^{(j)}(p - \rho(\omega)) + (-1)^j \delta^{(j)}(p + \rho(\omega))\big),$$

for some even distributions q_j, $q_j(\omega) = q_j(-\omega)$, on the sphere S^{n-1}.

Plan of proof of Theorem 1

1. Write down the condition that $\int_{\mathbb{R}} g(\omega, p)p^k dp$ is a polynomial of degree k in ω for each k.

2. Prove that those conditions imply that $\rho(\omega)^2$ must be a quadratic polynomial.

To compute

$$\int_{\mathbb{R}} g(\omega, p) p^k dp$$

we use for instance the fact that

$$\begin{aligned}\int_{\mathbb{R}} \delta'(p-\rho(\omega)) p^k dp &= -\int_{\mathbb{R}} \delta(p-\rho(\omega))\, k\, p^{k-1} dp \\ &= -k\,\rho(\omega)^{k-1}.\end{aligned}$$

Recall that

$$g(\omega, p) = \sum_{j=0}^{m-1} q_j(\omega) \big(\delta^{(j)}(p-\rho(\omega)) + (-1)^j \delta^{(j)}(p+\rho(\omega)) \big).$$

The range conditions therefore mean that there must exist polynomials p_0, p_2, p_4 etc., where $p_k(\omega)$ is homogeneous of degree k, such that (for instance if $m = 3$)

$$
\begin{aligned}
& q_0 = p_0 \\
& q_0\rho^2 + 2\,q_1\rho + 2\,q_2 = p_2 \\
& q_0\rho^4 + 4\,q_1\rho^3 + 4\cdot 3\,q_2\rho^2 = p_4 \\
& q_0\rho^6 + 6\,q_1\rho^5 + 6\cdot 5\,q_2\rho^4 = p_6 \\
& q_0\rho^8 + 8\,q_1\rho^7 + 8\cdot 7\,q_2\rho^6 = p_8 \\
& \ldots .
\end{aligned}
$$

Let us write this in matrix form.

$$
\begin{pmatrix}
1 & 0 & 0 \\
\rho^2 & 2\rho & 2 \\
\rho^4 & 4\rho^3 & 4\cdot 3\rho^2 \\
\rho^6 & 6\rho^5 & 6\cdot 5\rho^4 \\
\rho^8 & 7\rho^7 & 8\cdot 7\rho^6 \\
\rho^{10} & 10\rho^9 & 10\cdot 9\rho^8 \\
\ldots & \ldots & \ldots
\end{pmatrix}
\begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix}
=
\begin{pmatrix} p_0 \\ p_2 \\ p_4 \\ p_6 \\ p_8 \\ \ldots \end{pmatrix} .
$$

Recall that $\rho(\omega)$ is the supporting function of the set D. We want to prove that $\rho(\omega)^2$ must be a quadratic polynomial, because that is equivalent to ∂D being a quadric.

Forming suitable linear combinations of four of those equations we can eliminate the q-functions. This gives infinitely many equations of the form

$$
\begin{aligned}
\rho^6 p_0 - 3\rho^4 p_2 + 3\rho^2 p_4 &= p_6 \\
\rho^6 p_2 - 3\rho^4 p_4 + 3\rho^2 p_6 &= p_8 \\
\rho^6 p_4 - 3\rho^4 p_6 + 3\rho^2 p_8 &= p_{10} \\
\rho^6 p_6 - 3\rho^4 p_8 + 3\rho^2 p_{10} &= p_{12} \\
\cdots
\end{aligned}
$$

We now have only two kinds of functions of ω: the supporting function $\rho(\omega)$ and the polynomials $p_k(\omega)$. The only known fact is that $p_k(\omega)$ is a homogeneous polynomial in ω of degree k for every k.

Considering the first three equations as a linear system in the three "unknowns" ρ^2, ρ^4, and ρ^6, we can write those equations

(1) $$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} \rho^6 \\ -3\rho^4 \\ 3\rho^2 \end{pmatrix} = \begin{pmatrix} p_6 \\ p_8 \\ p_{10} \end{pmatrix}.$$

Provided the determinant of the matrix is different from zero, we can solve for instance ρ^2 from this system and obtain ρ^2 as a rational function

$$\rho(\omega)^2 = \frac{F(\omega)}{G(\omega)},$$

where $F(\omega)$ and $G(\omega)$ are polynomials, and

$$G(\omega) = \det \begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix}.$$

However, with very little additional effort we can do much better.

The following identities are trivial.

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_2 \\ p_4 \\ p_6 \end{pmatrix} \quad \text{and}$$

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_4 \\ p_6 \\ p_8 \end{pmatrix}.$$

Combining the linear system (1) with those two trivial equations we obtain the matrix equation

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} \begin{pmatrix} 0 & 0 & \rho^6 \\ 1 & 0 & -3\rho^4 \\ 0 & 1 & 3\rho^2 \end{pmatrix} = \begin{pmatrix} p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \\ p_6 & p_8 & p_{10} \end{pmatrix}.$$

The advantage with this equation is that it can be iterated. Setting

$$A = \begin{pmatrix} 0 & 0 & \rho^6 \\ 1 & 0 & -3\rho^4 \\ 0 & 1 & 3\rho^2 \end{pmatrix}$$

we have

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} A^2 = \begin{pmatrix} p_4 & p_6 & p_8 \\ p_6 & p_8 & p_{10} \\ p_8 & p_{10} & p_{12} \end{pmatrix}.$$

And more generally

$$\begin{pmatrix} p_0 & p_2 & p_4 \\ p_2 & p_4 & p_6 \\ p_4 & p_6 & p_8 \end{pmatrix} A^k = \begin{pmatrix} p_{2k} & p_{2k+2} & p_{2k+4} \\ p_{2k+2} & p_{2k+4} & p_{2k+6} \\ p_{2k+4} & p_{2k+6} & p_{2k+8} \end{pmatrix}$$

for every k. The determinant of A is $\rho(\omega)^6$. It follows that

$$G(\omega)\rho(\omega)^{6k} \quad \text{is a polynomial for every } k.$$

Since we already knew that $\rho(\omega)^2$ is a rational function, we can now conclude that $\rho(\omega)^2$ must be a polynomial (still assuming that $G(\omega)$ is not identically zero).

Therefore it remains only to prove

Lemma. If $q_{m-1} \neq 0$, then the $m \times m$ matrix

$$\begin{pmatrix} p_0 & p_2 & p_4 & \cdots & p_{m-2} \\ p_2 & p_4 & p_6 & \cdots & p_m \\ p_4 & p_6 & p_8 & \cdots & p_{m+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ p_{m-2} & p_m & p_{m+2} & \cdots & p_{2m-4} \end{pmatrix}$$

is non-singular.

This fact depends on the spectral properties of the matrix A.

Not necessarily symmetric D

An arbitrary distribution $g(\omega, p) = Rf(\omega, p)$ of order 0 can no longer be written

$$g(\omega, p) = q(\omega)\big(\delta(p - \rho(\omega)) + \delta(p + \rho(\omega))\big).$$

Instead we have to write

$$g(\omega, p) = q_0(\omega)\delta(p - \rho(\omega)) + q_0(-\omega)\delta(p + \rho(-\omega)).$$

Similar for higher order, but with different sign, for instance

$$q_1(\omega)\delta'(p - \rho(\omega)) - q_1(-\omega)\delta(p + \rho(-\omega)).$$

To shorten formulas write

$$\rho(\omega) = \rho, \quad \rho(-\omega) = \check{\rho}, \qquad q_j(\omega) = q_j, \quad q_j(-\omega) = \check{q}_j.$$

Then we get if $m = 3$

$$\begin{aligned} g(\omega, p) &= q_0\delta(p - \rho) + \check{q}_0\delta(p + \check{\rho}) \\ &= q_1\delta'(p - \rho) - \check{q}_1\delta'(p + \check{\rho}) \\ &= q_2\delta''(p - \rho) + \check{q}_2\delta''(p + \check{\rho}). \end{aligned}$$

Instead of the system

$$\begin{pmatrix} 1 & 0 & 0 \\ \rho^2 & 2\rho & 2 \\ \rho^4 & 4\rho^3 & 4\cdot 3\rho^2 \\ \rho^6 & 6\rho^5 & 6\cdot 5\rho^4 \\ \rho^8 & 7\rho^7 & 8\cdot 7\rho^6 \\ \rho^{10} & 10\rho^9 & 10\cdot 9\rho^8 \\ \dots & \dots & \dots \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} p_0 \\ p_2 \\ p_4 \\ p_6 \\ p_8 \\ \dots \end{pmatrix}$$

that we had before, we now get the system

$$
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
\rho & 1 & 0 & -\check{\rho} & -1 & 0 \\
\rho^2 & 2\rho & 2 & \check{\rho}^2 & 2\check{\rho} & 2 \\
\rho^3 & 3\rho^2 & 6\rho & -\check{\rho}^3 & -3\check{\rho}^2 & -6\check{\rho} \\
\rho^4 & 4\rho^3 & 12\rho^2 & \check{\rho}^4 & 4\check{\rho}^3 & 12\check{\rho}^2 \\
\rho^5 & 5\rho^4 & 20\rho^3 & -\check{\rho}^5 & -5\check{\rho}^4 & -20\check{\rho}^3 \\
\rho^6 & 6\rho^5 & 30\rho^4 & \check{\rho}^6 & 6\check{\rho}^5 & 30\check{\rho}^4 \\
\dots & \dots & \dots & \dots & \dots & \dots
\end{pmatrix}
\begin{pmatrix}
q_0 \\ q_1 \\ q_2 \\ \check{q}_0 \\ \check{q}_1 \\ \check{q}_2
\end{pmatrix}
=
\begin{pmatrix}
p_0 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ \dots
\end{pmatrix}.
$$

Eliminating the 6 densities $q_0, q_1, q_2, \check{q}_0, \check{q}_1, \check{q}_2$ as before we find that the successive 6-tuples from the infinite sequence $p_0, p_1, p_2, \ldots$ form an orbit of the matrix

$$A^t = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ r_0 & r_1 & r_2 & r_3 & r_4 & r_5 \end{pmatrix},$$

where the r_j are now the symmetric functions of degree $6 - j$ in the two eigenvalues ρ and $-\check{\rho}$. So for instance $r_0 = \det A = -\rho^3 \check{\rho}^3$.

In a different basis this matrix has the form

$$\begin{pmatrix} \rho & 1 & 0 & 0 & 0 & 0 \\ 0 & \rho & 2 & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 \\ 0 & 0 & 0 & \check{\rho} & 1 & 0 \\ 0 & 0 & 0 & 0 & \check{\rho} & 2 \\ 0 & 0 & 0 & 0 & 0 & \check{\rho} \end{pmatrix}.$$

This fact is used for the proof of the lemma above in this case.

Assuming that the rational function $\rho - \check{\rho}$ is not a polynomial I can deduce a contradiction using two expressions for the trace of A^k (just as I did using two expressions for $\det A^k$ above).

Hence $\rho - \check{\rho}$ is a polynomial. But ρ is homogeneous of degree 1 (as a function on $\mathbb{R}^n \setminus \{0\}$), hence $\rho - \check{\rho}$ must be a homogeneous first degree polynomial, that is, linear in ω.

But a translation of the coordinates adds a linear function to ρ, hence adds a linear function to $\rho - \check{\rho}$ (without changing $\rho + \check{\rho}$).

Therefore we can make a translation of coordinates so that $\rho - \check{\rho}$ vanishes. This means that ρ becomes symmetric, $\rho(\omega) = \rho(-\omega)$, and we are back to the case already treated.

A semi-local result

Theorem 4. Let D be open, convex, bounded, and symmetric, that is $D = -D$, let $x^0 \in \partial D$, and let ω^0 be one of the unit normals of a supporting plane L_0 to $\overline{D}$ at x^0. If there exists a distribution f with support in $\overline{D}$ and a translation invariant open neighborhood W of L_0, such that the restriction of the distribution Rf to W is supported on the set of supporting planes to D in W, then ∂D must be equal to the restriction of an ellipsoid in some neighborhood of $\pm x^0$.

A recent, somewhat related, result:

Theorem (Ilmavirta and Paternain, 2018). Let $D \subset \mathbb{R}^n$ be a bounded and strictly convex domain with smooth boundary. If there exists a function $f \in L^1(D)$ such that the integral of f over almost every line meeting D is equal to 1, then D is a ball.

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