Lines, broken lines and stars in tomography

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Acknowledgements

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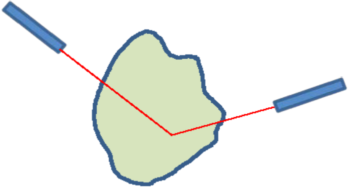
Outline

- Some Motivating Imaging Modalities
- Prior Work and Terminology
- Geometric Description
- Inversion of the Star Transform
- A Numerical Example

Single Scattering Optical Tomography (SSOT)



- Uses light, transmitted and scattered through an object, to determine the interior features of that object.
- If the object has moderate optical thickness it is reasonable to assume the majority of photons scatter once.
- Using collimated emitters/receivers one can measure the intensity of light scattered along various broken rays.
- Need to recover the spatially varying coefficients of light absorption μ_a and/or light scattering μ_s.

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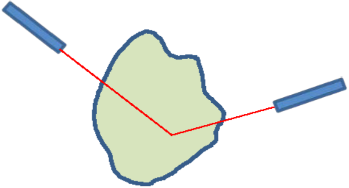

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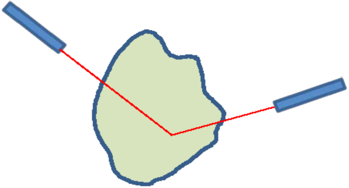

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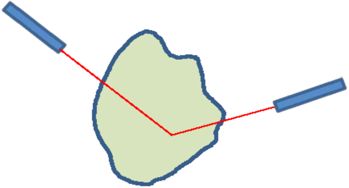

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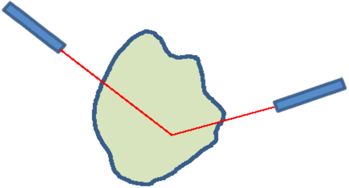

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Zhao, Schotland and Markel (2014)

Emitters and receivers collimated in several directions are used along the lines $z = 0$ and $z = L$.

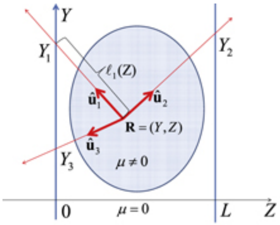
The signal measured by a detector in j-th direction due to an emitter in k-th direction is given by

$$W_{jk}(\mathbf{R}) = W_0\, S_{jk}\, \mu_s(\mathbf{R}) \exp\{-[l_j(\mathbf{R}) + l_k(\mathbf{R})]\},$$

where $l_k(\mathbf{R}) = \int_0^{\ell_k(Z)} \mu(\mathbf{R} + \hat{\mathbf{u}}_k \ell) d\ell, \quad \mu(y,z) = \mu_s(y,z) + \mu_a(y,z)$

W_0 is the constant power generated by the emitters and

S_{jk} is a constant that depends of the angle between $\hat{\mathbf{u}}_j$ and $\hat{\mathbf{u}}_k$.



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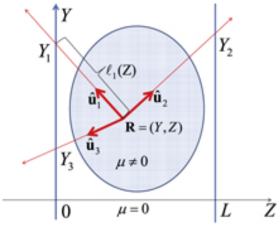

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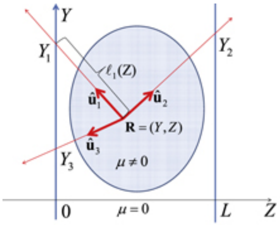

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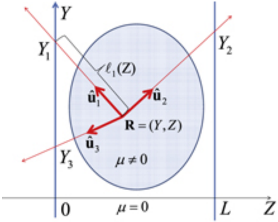

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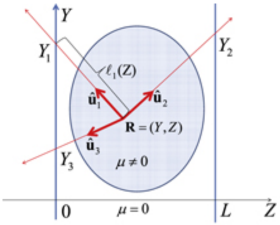

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If $\bar{\mu}_s$ is the background scattering coefficient, one can define the measured *data function* as

$$\phi_{jk}(\mathbf{R}) = \ln\left[\frac{W_{jk}(\mathbf{R})}{W_0\, S_{jk}}\,\bar{\mu}_s\right] = [\eta(\mathbf{R}) - I_j(\mathbf{R}) - I_k(\mathbf{R})]\,(1-\delta_{jk}),$$

where $\eta(\mathbf{R}) = \ln[\mu_s(\mathbf{R})/\bar{\mu}_s]$.

Excluding $\eta(\mathbf{R})$ from the above equations will substantially simplify the inverse problem.

If $\mu_s(\mathbf{R}) \equiv \bar{\mu}_s$ then $\eta(\mathbf{R}) = 0$ and the data function is equivalent to the broken ray transform of μ.

If $\mu_s(\mathbf{R})$ is not constant, one can use various linear combinations of the data function to eliminate $\eta(\mathbf{R})$.

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The star transform

E.g., in a "3-directional" geometry one can use a data function:

$$\Phi \equiv \phi_{12} - \phi_{13} = I_3 - I_2,$$

which is equivalent to the **signed** broken ray transform of μ.

Such an approach was used in the works of Florescu, Schotland, Markel (2009, '10, '11) and Katsevich, Krylov (2013, '15).

Notice, that in the above example we lost information about the integrals of the image function along rays of certain direction. The problem of excluding η, without excluding any ray integrals can be solved by finding coefficients c_{jk} such that

$$(i)\ \sum_{jk} c_{jk} = 0, \quad (ii)\ c_{kk} = 0, \quad (iii)\ c_{jk} = c_{kj}, \quad (iv)\ s_k = \sum_j c_{jk} \neq 0.$$

$$\text{Thus} \quad \Phi \equiv -\frac{1}{2}\sum_{j=1}^{K}\sum_{k=1}^{K} c_{jk}\phi_{jk} = \sum_{k=1}^{K} s_k I_k.$$

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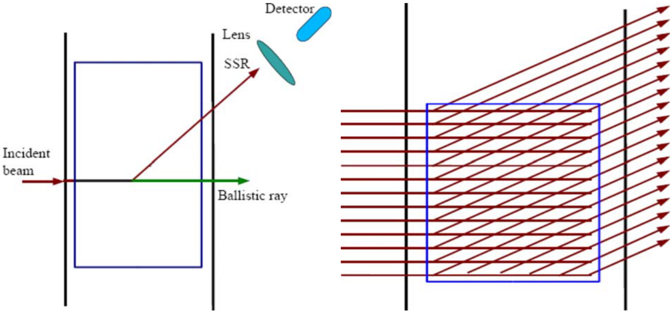
Florescu, Schotland and Markel (2009, 2010, 2011)

So if the scattering coefficient is known, then the reconstruction of the absorption coefficient is reduced to inversion of a generalized Radon transform integrating along the broken rays.



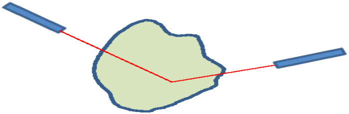
Broken Line (Broken Ray, V-line) and Conical Transforms

- L. Florescu, V. Markel, J. Schotland, F. Zhao
- P. Grangeat, M. Morvidone, M. Nguyen, R. Régnier, T. Truong, H. Zaidi
- A. Katsevich, R. Krylov
- P. Kuchment, F. Terzioglu
- V. Palamodov
- R. Gouia-Zarrad
- M. Courdurier, F. Monard, A. Osses and F. Romero
- D. Finch, B. Sherson

Broken Line (Broken Ray, V-line) and Conical Transforms

- M. Cree, P. Bones and R. Basko, G. Zeng, G. Gullberg
- M. Allmaras, D. Darrow, Y. Hristova, G. Kanschat, P. Kuchment
- M. Haltmeier
- C. Jung, S. Moon
- V. Maxim
- D. Schiefeneder
- M. Lassas, M. Salo, G. Uhlmann
- M. Hubenthal, J. Ilmavirta

V-line Transform (VLT) in 2D



Definition

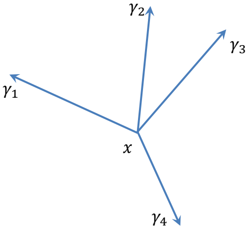
The V-line transform of a function $f(x, y)$ is defined as

$$Tf(\beta, t) = \int\limits_{L(\beta,t)} f\, ds, \tag{1}$$

where ds is the line measure along the V-line $L(\beta, t)$.

The problem of inversion of T is over-determined, so it is natural to consider a restriction of Tf to a two-dimensional set.

The Star Transform



Definition

The star transform $\mathcal{S}$ of f at $x \in \mathbb{R}^2$ is defined as:

$$\mathcal{S}f(x) = \sum_{i=1}^{m} \mathcal{X}_{\gamma_i} f(x) = \sum_{i=1}^{m} \int_0^\infty f(x + t\gamma_i)\, dt, \tag{2}$$

where each γ_i is a unit vectors and $\mathcal{X}_{\gamma_i}$ is the divergent beam transform in the direction of γ_i.

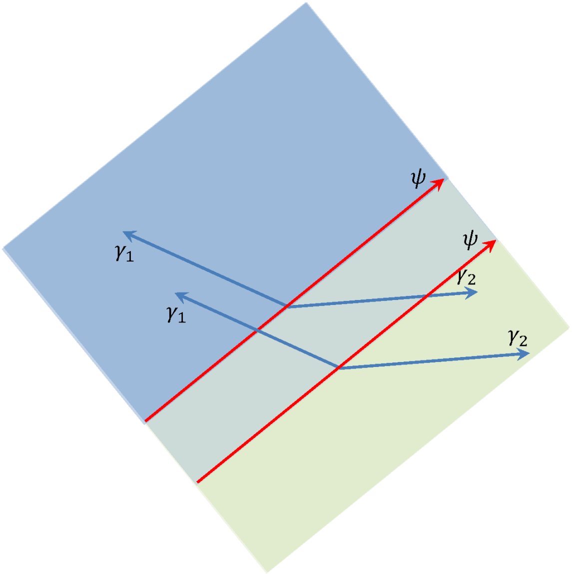
The Star Transform

- The star transform was first introduced in F. Zhao, J. Schotland and V. Markel “Inversion of the star transform” (2014) in relation to SSOT.
- The V-line transform is a special case of the star transform, with $m = 2$ and γ_1, γ_2 not parallel to each other.
- In SSOT the star transform allows reconstruction of the absorption and the scattering coefficients of the medium separately and simultaneously (from the same data).

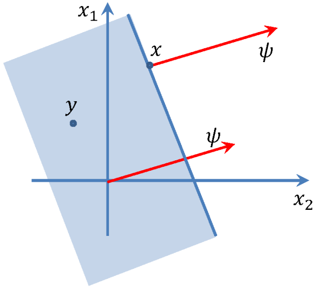
The Star Transform

- The star transform can also be used in single scattering X-ray tomography (SSXT). Here one can utilize scattered radiation which, in the case of the conventional X-ray tomography, is discarded.
- The method presented in the original paper FSM-2014 allows stable inversion of the star transform only for configurations involving odd number of rays.
- We present a new inversion method, which is based on simple geometric ideas and does not depend on the number of rays.

Geometric Description



Notations



Let $l(\psi, s) = \{x \in \mathbb{R}^2 |\, \langle x - s\psi, \psi \rangle = 0\}$ be the line normal to the vector ψ and of distance s from the origin.

Define $F(x) = \int_{\langle y,\psi\rangle \leq \langle x,\psi\rangle} f(y)\, d\mu$, and $F_\psi(s) = F(s\psi)$.

Composition of the Radon and Ray Transforms

Lemma

Assume that $f \in C_c(\mathbb{R}^2)$. *If* $\langle \psi, \gamma \rangle > 0$, *then*

$$\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = \frac{1}{\langle \psi, \gamma \rangle} F_{-\psi}(s),$$

and if $\langle \psi, \gamma \rangle < 0$, *then*

$$\mathcal{R}(\mathcal{X}_\gamma f)(\psi, s) = -\frac{1}{\langle \psi, \gamma \rangle} F_{\psi}(s).$$

Inversion of the Star Transform

Theorem

Let $\mathcal{S} = \sum_{i=1}^{m} \mathcal{X}_{\gamma_i}$ *be the star transform and let*

$$q(\psi) = \frac{-1}{\sum_{i=1}^{m} \frac{1}{\langle \psi, \gamma_i \rangle}}.$$

Then the following is true for any ψ *in the domain of* q

$$\mathcal{R}f(\psi, s) = q(\psi) \frac{d}{ds} \mathcal{R}(\mathcal{S}f)(\psi, s).$$

Hence, if q *is defined almost everywhere, we can apply* $\mathcal{R}^{-1}$ *to recover* f.

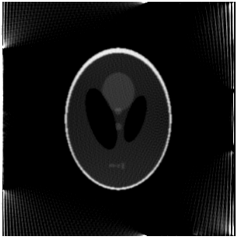
Inversion of the V-line Transform

Corollary

An inversion formula for the V-line transform with ray directions γ_1, γ_2 *is given by*

$$f = \mathcal{R}^{-1}\left(\frac{-\langle\psi,\gamma_1\rangle\langle\psi,\gamma_2\rangle}{\langle\psi,\gamma_1\rangle+\langle\psi,\gamma_2\rangle}\frac{d}{ds}\mathcal{R}(\mathcal{S}f)(\psi,s)\right).$$

A Numerical Example



Reconstruction using the star transform with directions $\gamma_1 = (1, 0)$, $\gamma_2 = (cos(3\pi/4), sin(3\pi/4))$, $\gamma_3 = (cos(5\pi/4), sin(5\pi/4))$

Thanks for Your Attention!